ASYMPTOTIC ANALYSIS OF A STATIONARY CRACK IN A DUCTILE FUNCTIONALLY GRADED MATERIAL

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Summary The dominant and higher-order asymptotic stress and displacement fields surrounding a stationary crack in a ductile functionally graded material subjected to anti-plane shear loading are derived. It is shown that the leading (most singular) term in the asymptotic expansion is the same in the graded material as in the homogeneous one. The second term in series may differ from that of the homogeneous case depending on the particular material property variation.

INTRODUCTION

Many naturally occurring engineering materials have properties that vary continuously and smoothly with position. Examples include animal bones, teeth and wood. Engineering materials with inhomogeneous material properties are used in order to design efficient structures. For the purpose of this article, a *functionally graded material* is one, in which the material properties such as Young's modulus, density or yield stress are a function of spatial position and the functional form of the variations are known. Functionally graded materials have a broad range of applications as in thermal barrier coatings in gas turbines, artificial dental implants, bearing surfaces in MEMS devices and ballistic impact resistant structures. Functionally graded materials can be formed in a variety of ways, for example a thermal barrier coating is formed by depositing a porous ceramic coating on a metallic substrate (bond coat). The material properties of such a a graded system varies from that of a pure metal on one side to that of a ceramic on the other and hence will have the advantages of both a metal and ceramic at the same time. Fracture behavior of graded materials is not fully understood which limits their usage. Static (Eischen, 1987) and dynamic (Parameswaran and Shukla, 1999) analysis of cracks in inhomogeneous materials do exist in the literature, but they all ignore the effect of inelastic deformations even when the material is ductile. In this paper, we focus our attention on the asymptotic solution of a stationary crack in a ductile, inhomogeneous material subjected to an anti-plane shear loading. The analysis is developed in the lines of the HRR solution (Hutchinson, 1968; Rice and Rosengren, 1968) and the elastic deformations are assumed to be negligibly small compared to the plastic ones. A variable separable solution is assumed to exist near the crack tip, even when the material properties vary with spatial position. As in the elastic case (Eischen, 1987; Parameswaran and Shukla, 1999), we see that the first-term (the most singular term) of the asymptotic solution remains unaffected by the material inhomogeneity even in the plastic case. In this article we explore whether the second-term is affected by the material property variation.

FORMULATION

A semi-infinite stationary crack embedded in a functionally graded material subjected to a far field mode 3 loading is considered. The analysis is performed in the polar coordinates choosing the crack tip as the origin. The material is assumed to follow the infinitesimal J_2-deformation theory along-with the Ramberg-Osgood power law hardening model. The shear stresses in the anti-plane case can be written as

$$\sigma_{\gamma z} = G\left(r,\theta\right)\left[2\epsilon_{\beta z}\epsilon_{\beta z}/3\right]^{\frac{1}{n}-1}\epsilon_{\gamma z},\ \gamma = r,\theta, \tag{1}$$

with $G = 2\sigma_o\left[\alpha\epsilon_o\right]^{-1/n}/3$, where σ_o and ϵ_o are the stress and strain at initial yield respectively and α is a non-dimensional parameter. For mathematical simplicity we restrict the hardening index n to be a constant, while allowing the other material properties in the above equation to vary with position. Traction free boundary condition is enforced on the crack faces in order to obtain a solution ($\sigma_{\theta z}(r,\pm\pi) = 0$).

LEADING ORDER TERM

The leading order solution is obtained by assuming a variable separable one-term expansion for the out of plane displacement as

$$\frac{w\left(r,\theta\right)}{r_P} \approx A\left(\frac{r}{r_P}\right)^{p+1} f\left(\theta\right)\ \text{as}\ r\to 0, \tag{2}$$

where r_P is the radius of the plastic zone, A is a non-dimensional amplification factor and p is the strain singularity, which must be greater than -1 to ensure a bounded displacement field. The governing non-linear ordinary differential equation for $f(\theta)$ is obtained by using the strain-displacement, stress-strain and stress equilibrium equations, which can be written as

$$G\phi_1 + \frac{r_P}{r_F}\frac{r}{r_P}\left[\frac{\partial G}{\partial r}\phi_{2r} + \frac{\partial G}{\partial\theta}\phi_{2\theta}\right] = 0, \tag{3}$$

here r_F denotes the intrinsic material length scale associated with the material gradient. In the above equation ϕ_1, ϕ_{2r} and $\phi_{2\theta}$ are functions of the angular function f and its derivatives. When the material gradient is absent the above system of equations reduce to $\phi_1 = 0$ and the solution for the homogeneous material is recovered. The material property G is required to be finite and strictly positive at the crack tip and hence the solution to the leading order term in the case of a graded material is identical to that of a homogeneous material. This result is expected and follows the trend observed in elastic analyses of graded materials (Eischen, 1987).

HIGHER ORDER TERM

As seen in the previous section, the material property gradient does not affect the solution to the leading order term. Here, we explore the possibility whether the second (less singular) term is affected by the material gradient.

Homogeneous case

Now the displacement is assumed to be given by a two term expansion as

$$\frac{w(r,\theta)}{r_P} \approx A\left(\frac{r}{r_P}\right)^{p+1} f(\theta) + B\left(\frac{r}{r_P}\right)^{q+1} g(\theta), \tag{4}$$

where A and B are non-dimensional amplitude factors, p and q are the exponents for the leading and second terms, respectively (with $q > p$). A set of governing non-linear ordinary differential equations are obtained for f and g as

$$Ar^{3p-1}\phi_1 = 0, Br^{2p+q+1}\phi_3 = 0 \text{ and } \left.\frac{df}{d\theta}\right|_{\pm\pi} = \left.\frac{dg}{d\theta}\right|_{\pm\pi} = 0. \tag{5}$$

In the above equation ϕ_3 is a function of both the angular functions f and g and their derivatives. The solution to the first of the above equations is identical to the solution obtained earlier for the leading order term. With the knowledge of p and f a solution for q (eigen value) and g (eigen vector) can be obtained by solving $\phi_3 = 0$. The amplitude factor B is left undetermined by this asymptotic analysis. The exponent q obtained for the homogeneous material is denoted as q_h and forms as the upper bound for the second term exponent of a graded material as shown in the next section.

Graded material

A two term expansion for the out of plane displacement described in the previous section is used even in the case of the graded material. In the homogeneous case the material property G was independent of the spatial position. In the graded material we assume the material property to be

$$G(r,\theta) = G_o + \left[\frac{r}{r_F}\Gamma(\theta)\right]^c G_1. \tag{6}$$

The above type of power-law material property variation is most generic and can describe any functional form of property variation, such as linear, exponential etc., by setting $c = 1$ (Taylor series expansion). By choosing an appropriate trigonometric function for $\Gamma(\theta)$ the material property can be described along radial lines, on the crack line and perpendicular to the crack plane. The boundary value problem for the higher order term for a graded material is now given by

$$\phi_1 = 0, \phi_3 + \left(\frac{r}{r_P}\right)^{(c+p)-q} \underbrace{c\left(\frac{G_1}{G_o}\right)\left(\frac{r_P}{r_F}\right)^c \frac{A}{B}}_{m} \left[\phi_{2r} + \frac{1}{\Gamma}\frac{d\Gamma}{d\theta}\phi_{2\theta}\right] = 0, \text{ and } \left.\frac{df}{d\theta}\right|_{\pm\pi} = \left.\frac{dg}{d\theta}\right|_{\pm\pi} = 0. \tag{7}$$

It can be seen from the above equation that the higher order term will or not be affected by the material property variation depending on the value of $(c+p) - q$. When $q > c + p$ the first term which is the most singular term is the only possible solution. When $q < c + p$ the asymptotic solution for a graded material is identical to that of a homogeneous material upto and including the second term. The material property gradient is going to alter the asymptotic solution only when $q = c + p$. In this case the exponent of the second term is completely determined and the multiplying factor in (7) $m = c(G_1 G_o)(r_P/r_F)^c(A/B)$ becomes the eigen value of the boundary value problem. It is interesting to note that the amplification factor B which was left undetermined by the asymptotic analysis in the homogeneous case is now fully determined. Our analysis has shown that a variable separable solution is not possible when $q \geq q_h$, which specifies an upper bound for the material exponent as $c_{max}(n) = q_h - p$.

References

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