THREE-DIMENSIONAL GLOBAL MODES IN SPATIALLY VARYING RAYLEIGH–BÉNARD–POISEUILLE CONVECTION

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Summary The Rayleigh–Bénard–Poiseuille convection with spatially varying difference of temperatures on the upper and lower plates is the archetype of systems sustaining the growth of synchronised global modes. An analytical construction method and a selection criterion for the most unstable linear instability is proposed in cases of slowly varying systems. The results are in very good agreement with direct numerical simulations.

This study focuses on the instabilities in a laminar Poiseuille flow cooled from its upper plate and heated from its lower one, the latter of which presents a two-dimensional bump of temperature. This flow is representative of systems exhibiting temporally synchronised, or global, instabilities despite a spatial variation of the energy flux transferred to these perturbations — flux imposed by the difference between the temperatures of the lower and upper plates in this case of Rayleigh–Bénard–Poiseuille convection. This tuning to a self-selected frequency cannot be accounted for by a local criterion straightforwardly extended from the homogeneous system. Hence a global selection criterion, taking the inhomogeneity into account, has to be sought. Whereas an arbitrary spatial variation of the basic state is usually only accessible through experimental or numerical stability analysis, the description of slowly varying systems can be coped with analytically. Slow variation indeed allows us to consider local stability properties, such as convective and absolute instability, as the lowest order of a description taking the inhomogeneity into account by means of a Wentzel–Kramers–Brillouin–Jeffreys (WKBJ) expansion. Beyond the local absolute instability threshold, the group velocity of some propagative waves solutions of the local homogeneous stability problem can vanish at some points of the physical domain, called turning points, where the WKBJ expansion breaks down. Our present work aims at extending to two-dimensional situations which present an “island” of instability, i.e. the two-dimensional bump of temperature, the analytical construction of a global mode governed by a double turning point located at the maximum of the local growth rate . The analytical flows and critical conditions obtained by this method are then compared with direct numerical simulations of the Navier–Stokes equations under the Boussinesq approximation.

GLOBAL MODE CONSTRUCTION AND SELECTION

WKBJ expansion

In the inhomogeneous Rayleigh–Bénard–Poiseuille flow, the slow and rapid horizontal variations are separated by the use of slow coordinates $X = \varepsilon x$ and $Y = \varepsilon y$, where ε is a small parameter. The basic steady state is sought as an expansion in powers of ε. The WKBJ expansion of the perturbation is expressed as

$$\boldsymbol{v} = \left(\boldsymbol{v}_0 \left(X, Y, z\right) + \varepsilon \boldsymbol{v}_1 \left(X, Y, z\right) + O\left(\varepsilon^2\right)\right) \exp\left(\frac{\mathrm{i}}{\varepsilon}\Phi\left(X, Y\right) - \mathrm{i}\omega t\right) + \text{c.c.}, \tag{1}$$

with the complex frequency $\omega = \omega_0 + \varepsilon\omega_1 + O\left(\varepsilon^2\right)$ and wavevector $\boldsymbol{k}\left(X, Y\right) = \boldsymbol{\nabla}\Phi\left(X, Y\right)$. At order $O\left(\varepsilon^0\right)$ of the linearised dynamics equations, the perturbation is sought as a solution of the homogeneous local problem — parameterised by the local Rayleigh number $\mathcal{R}\left(X, Y\right)$ with the frequency ω_0 and the wavevector $\boldsymbol{k}\left(X, Y\right)$ — modulated by an amplitude $A\left(X, Y\right)$. As a consequence the dispersion relation, expressed as

$$\omega\left(\partial_X \Phi, \partial_Y \Phi, \mathcal{R}\left(X, Y\right)\right) = \omega_0, \tag{2}$$

is satisfied by $\Phi\left(X, Y\right)$. At order $O\left(\varepsilon^1\right)$ the solvability condition for the perturbation leads to the amplitude equation satisfied by $A\left(X, Y\right)$:

$$\partial_X A\, \partial_{k_x}\omega + \partial_Y A\, \partial_{k_y}\omega + A\left(-\mathrm{i}\omega_1 + \Gamma\left(X, Y; \omega_0\right)\right) = 0, \tag{3}$$

where $\Gamma\left(X, Y; \omega_0\right)$ gathers terms due to the spatial dependences of the basic state, the wavevector and $\mathcal{R}$. The integration of $\Phi\left(X, Y\right)$ and $A\left(X, Y\right)$, which results in the complete first order of the WKBJ expansion, can be done along the — common — characteristics of equations (2) and (3), whose tangential vector is the group velocity $\left(\partial_{k_x}\omega, \partial_{k_y}\omega\right)$, where the values of ω_0 and ω_1 in these equations must be known and a gauge choice for Φ and A must be given. This can be achieved by means of a two-dimensional double turning point, where $\partial_{k_x}^{\text{t.p.}}\omega = \partial_{k_y}^{\text{t.p.}}\omega = 0$ and $\partial_X^{\text{t.p.}}\omega = \partial_Y^{\text{t.p.}}\omega = 0$.

Turning point region and selection criterion

Since the WKBJ approximation breaks down at a turning point the perturbation is sought as an expansion in powers of $\varepsilon^{1/2}$. The homogeneous problem recovered at order $O\left(\varepsilon^0\right)$ and the condition $\partial_{k_x}^{\text{t.p.}}\omega = \partial_{k_y}^{\text{t.p.}}\omega = 0$ imposes the solution

$\widehat{\boldsymbol{v}_0}^{\text{t.p.}}(z)$, associated with the frequency $\omega_0^{\text{t.p.}}$ and the wavevector $\left(k_x^{\text{t.p.}}, k_y^{\text{t.p.}}\right)$, of this eigenproblem expressed at the turning point. The condition $\partial_X^{\text{t.p.}}\omega = \partial_Y^{\text{t.p.}}\omega = 0$ imposes the double turning point to be located at the maximum of $\mathcal{R}(x,y)$. The solvability condition at order $O\left(\varepsilon^1\right)$ yields the amplitude of the first order of the perturbation. This first order of the perturbation is then given by:

$$\boldsymbol{v}_0^{\text{inner}}(x,y,z,t) = \widehat{\boldsymbol{v}_0}^{\text{t.p.}}(z)\exp\left(-\frac{\varepsilon\alpha}{2}x^2 - \frac{\varepsilon\beta}{2}y^2 - \varepsilon\delta xy\right)\exp\left(\mathrm{i}\left(k_x^{\text{t}}x + k_y^{\text{t}}y\right) - \mathrm{i}\left(\omega_0^{\text{t.p.}} + \varepsilon\omega_1\right)t\right), \quad (4)$$

where the coefficients $\varepsilon\alpha$, $\varepsilon\beta$ and $\varepsilon\delta$ and the correction of the complex frequency $\varepsilon\omega_1$ are deduced from the dispersion relation and the $\mathcal{R}(x,y)$ function expressed at the turning point. Furthermore, the first order of the global mode in the turning point region (4) provides a gauge choice for Φ and A.

COMPARISON WITH NUMERICAL SIMULATIONS

For functions $\mathcal{R}(x,y)$ as in figure 1(a), the leading order of the inner expansion of the most unstable mode (4) and its critical conditions flowing from the evaluation of $\omega_0^{\text{t.p.}} + \varepsilon\omega_1$ are compared with the results of a spectral method direct numerical simulations. Rolls orientated transversely with respect to the mean flow, predicted analytically (figure 1(b)) and observed numerically (figure 1(c)), and the very good agreement for their critical conditions underline the relevancy of the analytical selection criterion.

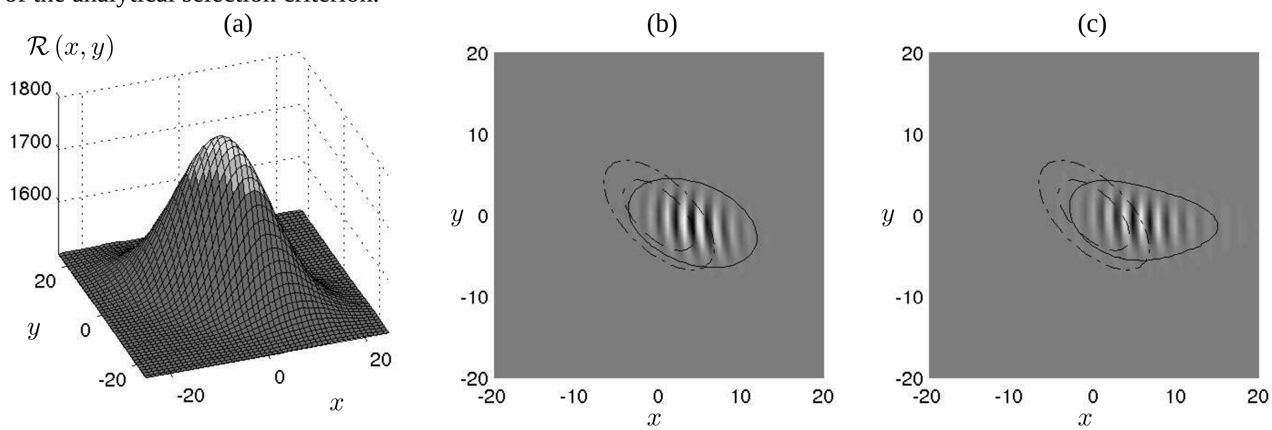

Figure 1. (a): bump of Rayleigh number $\mathcal{R}(x,y)$ with, for increasingly lighter shades of gray, the stable, convectively unstable and absolutely unstable regions obtained for a mean flow along the x-direction at Reynolds number $R = 0.40$ and a Prandtl number $P = 7$. Comparison between the vertical velocity components of the convection rolls of the analytical approximation (b) and direct numerical simulation at time $t = 125$ (c), with the convectively unstable/stable boundary (dashed-dotted lines), the absolutely/convectively unstable boundary (dashed lines) and the contour where the amplitude of the perturbation reaches five percent of its maximum (solid lines).

Furthermore, the numerical simulations shed light on the nonlinear behaviour of the instabilities. They distinguish situations where the linear mode saturates and preserves its maximum close to the downstream convectively unstable/stable boundary (figure 2(a)) from situations where a front forms at the upstream convectively/absolutely unstable boundary (figure 2(b)), which raise the question of the two-dimensional dynamics of a front.

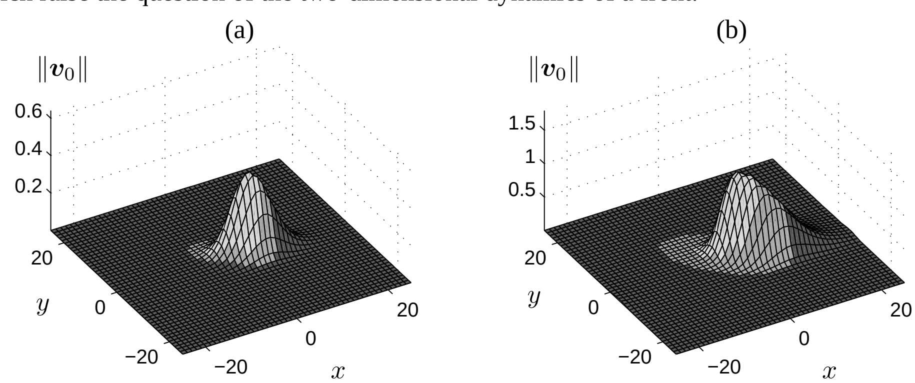

Figure 2. Nonlinear evolution of the amplitude of the instability obtained by numerical simulation for Reynolds numbers $R = 0.4$ (a) and $R = 0.38$ (b), with superimposed local stability properties as in figure 1(a).