Stability of compressible elastic blocks

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Introduction

In recent work Chen and Haughton [1–4] have developed a method to study the full non–linear stability of inhomogeneous deformations based on the second variation of the energy. The method is quite general and ultimately involves the solution of a quadratically non–linear system of ordinary differential equations. For two dimensional problems, such as the one we consider here, we have a third order system. The corresponding bifurcation criterion for two dimensional problems involves the solutions of a linear fourth order system. Here we consider the simple problem of the equi-biaxial loading of a cube which is held fixed in the third direction. The simplicity of the problem allows us to make considerable analytic progress. The corresponding problem for incompressible materials has been considered by several authors. A review can be found in Ogden [5].

Basic Equations

We consider the biaxial plane strain deformation of a compressible elastic cube

$$0 \leq X \leq A, \quad 0 \leq Y \leq A, \quad 0 \leq Z \leq A, \tag{1}$$

where (X, Y, Z) are the material cartesian coordinates. The body is comprised of a homogeneous, isotropic, and compressible elastic material in the reference configuration. It is assumed to undergo the homogeneous deformation

$$x = \lambda_1 X,\ y = \lambda_2 Y,\ z = \lambda_3 Z, \tag{2}$$

where (x, y, z) are the spatial cartesian coordinates, and λ_i, $i = 1, 2, 3$ are the constant, positive, principal stretches. We suppose that the deformation is accomplished by equal loads T applied to the sides $X = 0, A$ and $Y = 0, A$ with the remaining sides $Z = 0, A$ held a fixed distance apart. The axial stretch λ_3 can then be regarded as a prescribed parameter for this problem. The equilibrium equations are automatically satisfied for this homogeneous deformation and the dead loading T is given by

$$T = W_1 = W_2, \tag{3}$$

where $W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function of the material and subscripts indicate partial differentiation, $W_i = \partial W/\partial\lambda_i$. The second equation in (3) gives either the trivial solution

$$\lambda_1 = \lambda_2, \tag{4}$$

or,

$$W_1 = W_2, \quad \lambda_1 \neq \lambda_2. \tag{5}$$

Taking the limit as $\lambda_2 \to \lambda_1 = \lambda$, say, in (5), we obtain the bifurcation point where the trivial and non-trivial solutions cross. This is given by

$$W_{11}(\lambda, \lambda, \lambda_3) = W_{12}(\lambda, \lambda, \lambda_3). \tag{6}$$

The non-trivial solution (5) can be regarded as an implicit equation for $\lambda_2(\lambda_1)$. Using this interpretation we can show that bifurcation points coincide with turning points of the load T.

Stability Analysis

For stability we require that the second variation of the energy is non negative. It is easily shown that

$$\ddot{E} = \int_\Omega \{\nabla\dot{\mathbf{x}} \cdot W_{\mathbf{FF}}[\nabla\dot{\mathbf{x}}]\} dV\ , \tag{7}$$

where $\dot{\mathbf{x}}$ is the variation in $\mathbf{x}$. We write $\nabla\dot{\mathbf{x}}$ as

$$\nabla\dot{\mathbf{x}} = \begin{pmatrix} u_X & u_Y & 0 \\ v_X & v_Y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

where a subscript denotes the partial derivative. The non-zero components of $W_{\mathbf{FF}}$ are given by

$$(W_{\mathbf{FF}})_{iijj} = W_{ij}\ , \qquad i, j,\ \text{no sum}, \tag{9}$$

$$(W_{\mathbf{FF}})_{ijij} = \frac{\lambda_i W_i - \lambda_j W_j}{\lambda_i^2 - \lambda_j^2}\,, \quad (W_{\mathbf{FF}})_{ijji} = \frac{\lambda_j W_i - \lambda_i W_j}{\lambda_i^2 - \lambda_j^2}\,, \qquad i \neq j\,, \qquad \lambda_i \neq \lambda_j\,, \text{ no sum.} \tag{10}$$

The integrand in (7) is then

$$\nabla\dot{\mathbf{x}} \cdot W_{\mathbf{FF}}[\nabla\dot{\mathbf{x}}] = W_{11}u_X^2 + W_{22}v_Y^2 + 2W_{12}u_X v_Y + W_{1212}(u_Y^2 + v_X^2) + 2W_{1221}u_Y v_X . \tag{11}$$

From this we deduce that the deformation is stable if and only if

$$W_{11} > 0\,, \quad W_{22} > 0\,, \quad W_{11}W_{22} - W_{12}^2 > 0\,, \quad W_{1212}^2 - W_{1221}^2 > 0\,, \tag{12}$$

since $W_{1212} > 0$ by the Baker–Ericksen inequalities.

Alternative Stability Analysis

The integrand in (7) involves two perturbation functions u and v. First we expand these functions into Fourier series in Y. We can show that we get

$$u(X,Y) = \sum_{n=1}^{\infty} f_n(X) \sin\frac{n\pi Y}{A} = \sum_{n=1}^{\infty} f_n(X)\sin\alpha Y, \quad v(X,Y) = \sum_{n=1}^{\infty} g_n(X)\cos\alpha Y. \tag{13}$$

Substituting (13) into (7), integrating with respect to Y and then applying the method outlined in [1–4] we have the alternative (but equivalent) stability criterion: First we solve the system of equations

$$\begin{aligned} y_1' &= \frac{(W_{1212}^2 - W_{1221}^2)}{W_{1212}} - \frac{y_1^2}{W_{11}} + \frac{y_2(2\,W_{1221} - y_2)}{W_{1212}}, \\ y_2' &= -\frac{y_1(W_{12} + y_2)}{W_{11}} + \frac{y_3(W_{1221} - y_2)}{W_{1212}}, \\ y_3' &= \frac{W_{11}W_{22} - W_{12}^2}{W_{11}} - \frac{y_2(2\,W_{12} + y_2)}{W_{11}} + \frac{y_3^2}{W_{1212}}, \quad W_{11} \neq 0\,, \end{aligned} \tag{14}$$

with

$$y_i(0) = 0, \quad i = 1,2,3. \tag{15}$$

For stability of the deformed body it is necessary and sufficient that

$$y_1(n\pi) > 0\,, \quad y_3(n\pi) > 0\,, \quad y_1(n\pi)y_3(n\pi) - y_2(n\pi)^2 > 0\,. \tag{16}$$

The problem of assessing the full non–linear stability of a body in a particular configuration then reduces to that of solving the initial value problem (14) with (15) and evaluating (16). We can determine any change in stability to within the accuracy of the numerical methods used. In particular we note that $y_i \equiv 0$ is a solution when

$$W_{1212}^2 - W_{1221}^2 = 0\,, \quad \text{and} \quad W_{11}W_{22} - W_{12}^2 = 0, \tag{17}$$

and this can be shown to be the case at the bifurcation point (6). We can make little analytic progress with (14) in general and so we mainly look at some specific examples to compare numerical results from (14–16) with the known results obtained in (12).

References

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