MULTI-SCALING APPROACH IN THE MECHANICS OF DISORDERED MATERIALS

Alberto Carpinteri, Simone Puzzi
Department of Structural Engineering and Geotechnics, Polytechnic of Turin, Italy

Summary The present paper reports the major results of the fractal approach to scaling of mechanical properties in solid mechanics. Among them, it will be put forward the size-independent *fractal cohesive crack model* and the definition of fractal quantities (hence with non-integer dimensions) which are the true material size-independent properties. Particular attention is paid to the fractal strain, defined by means of fractional calculus, and to the Principle of Virtual Work for fractal media.

One of the most important topics in solid mechanics is the study of the so-called *size effects*, whose importance has been widely recognised during the last decades. The first to address this topic was Leonardo da Vinci, in the 15th century, in his considerations about the strength of ropes of different length. Size-scale effects on the strength of materials were evidenced also by Galileo Galilei, about two centuries later, and by Mariotte. Despite of these enlightened, pioneering contributions, well ahead of their time, it took more or less five hundred years, before the importance of scaling, which cannot be predicted by classical continuum mechanics, clearly emerged. Probably, the most important step forward in the history of scaling of strength was given by Griffith in 1921, as he posed the basis of the linear elastic fracture mechanics (LEFM). And it took about fifty years more before the size-scale effects were recognised not only on strength, but also on the fracture energy, when arose the problem of designing large concrete structures, for which the gap between the scale of the structure and that of laboratory specimens is very large. Since the 1970's, the interest to scaling inside the scientific community has continuously grown and many valuable contributions have been given to the development of this nowadays fast growing, broad branch of solid mechanics. In the present century, the new challenge is at the opposite side: down-scaling, i.e. the size-scale effects when the structural scale is reduced and approaches the micro- (or even the nano-) scale. This strong interest is clearly emerging for various high-tech materials and applications. Several of these novel materials can be successfully represented and modelled as heterogeneous and disordered. For this reason we are convinced that the fractal approach, originally developed for this kind of materials, and the only one capable to encompass in a unified framework, through the definition of the Principle of Virtual Work for fractal media, the scaling on strength, fracture energy and critical strain, will succeed in capturing the essentials also of down-scaling.

The size-independent fractal cohesive crack model

The cohesive crack model is able to simulate tests where high stress gradients are present, e.g. tests on pre-notched specimens. Particularly, the model captures the ductile-brittle transition occurring by increasing the size of the structure. On the other hand, uniaxial tensile tests on dog-bone shaped specimens [3] have shown that the three material parameters defining the cohesive law are not size independent: particularly, increasing the specimen size, the tensile strength σ_u tends to decrease, whilst the fracture energy G_F and the critical displacement w_c tend to increase. These *size effects* can not be predicted by the cohesive model. In other words, the parameters of the cohesive crack model introduced by Barenblatt and Hillerborg [1] depend on the structural size. In order to overcome the original cohesive crack model drawbacks, a *scale-independent* (fractal) *cohesive crack model* has been proposed recently by the first Author. This model is based on the assumption of a fractal-like damage localisation in a *fractal band* at the mesostructural level. The fractal nature of the damage process allows us to explain the size effects on σ_u, G_F and w_c as well as the rising of the cohesive law tail observed in [5].

As fundamental hypothesis, bore out by experimental evidence, we define fractal geometries for both the resistant cross section at maximum load (fig. 1b) and the dissipation domain (fig. 2b) [4]. Hence we can compute the maximum load F, the critical displacement w_c and the total dissipated energy W as:

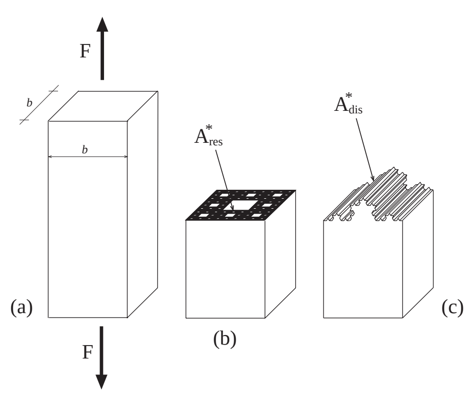

Figure 1: Tension specimen (a). Fractal localization of the stress upon the resistant cross section (b) and of the energy dissipation upon crack surface (c).

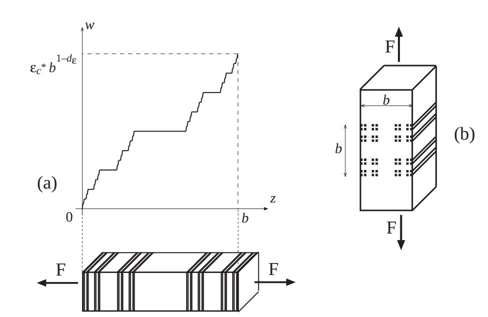

Figure 2: Fractal localization of the strain (a) and of the energy dissipation inside the damaged band (b).

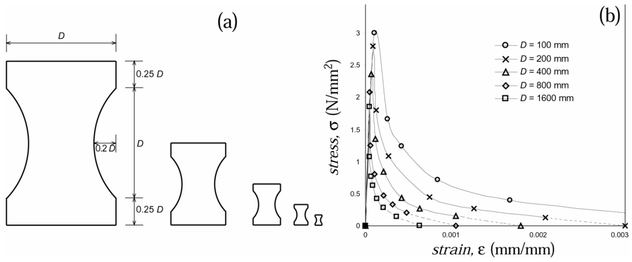

Figure 3: Tensile tests on dog-bone shaped specimens (a) by van Vliet [3]: stress-strain diagrams (b), cohesive law diagrams (c), fractal cohesive law diagrams (d). The dashed lines refer to extrapolated values.

$$F = \sigma_u A_0 = \sigma_u^* A_{\text{res}}^* , \qquad w_c = \varepsilon_c b = \varepsilon_c^* b^{1-d_\varepsilon} , \qquad W = G_F A_0 = G_F^* A_{\text{dis}}^* . \qquad (1)$$

These quantities are size-dependent. The true scale-independent quantities are the right hand side ones, i.e. the *fractal strength* σ_u^*, the *fractal critical strain* ε_c^* and the *fractal fracture energy* $\mathcal{G}_F^*$. They show non-integer physical dimensions: $[F][L]^{-(2-d\sigma)}$ for σ_u^*, $[L]^{d\varepsilon}$ for w_c^*, and $[FL][L]^{-(2+dG)}$ for G_F^*. Because of the measure of the resistant cross section A_{res}^* and the dissipation domain A_{dis}^*, from (1) the scaling laws for strength, critical displacement and fracture energy can be obtained:

$$\sigma_u = \sigma_u^* \, b^{-d_\sigma} , \qquad w_c = \varepsilon_c^* \, b^{1-d_\varepsilon} , \qquad G_F = G_F^* b^{+d_G} . \qquad (2)$$

Eqns (2) provide size effects in perfect agreement with the trends arising from experiments. The third scaling law can be derived also from the assumption, experimentally verified, of invasive fractal crack surface (fig. 1c). Moreover, we know that the sum of the three scaling exponents is always equal to one : $d_\sigma + d_\varepsilon + d_G = 1$ [4]. According to these definitions, we call the σ^*-ε^* diagram the *fractal or scale-independent cohesive law*. Contrarily to the classical cohesive law, which is experimentally sensitive to the structural size, this curve is an exclusive property of the material since it is able to capture the fractal nature of the damage process. The area below the softening fractal stress-strain diagram represents now the fractal fracture energy G_F^*.

The model has been applied to some data obtained by van Mier & van Vliet [3] from direct tensile tests under rotating boundary conditions and in a scale range 1:32 (fig. 3a). Interpreting the size effect on the ultimate stress and on the fracture energy by means of fractals and fitting the experimental results, the values $d_\sigma = 0.18$ and $d_G = 0.09$ are calculated. Some of the σ–ε and the σ-*w* (the cohesive law) diagrams are reported respectively in fig. 3b and 3c, where *w* is the displacement localized in the damage band, obtained by subtracting, from the total one, the displacement due to elastic and inelastic pre-peak deformation. Note how the tail of the cohesive law rises while its peak decreases increasing the size. The fundamental relationship between the scaling exponents yields to find for d_ε a value equal to 0.73, so that the fractal cohesive laws can be plotted in fig. 3d. As expected, all the curves related to different sizes tend to merge in a unique, scale-independent cohesive law.

Local fractional calculus and the Principle of Virtual Work for fractal media

The most recent work [5,6] is devoted to the possible application of the fractional calculus (i.e. integrals and derivatives of non-integer order) to the study of fractal phenomena, which cannot be handled by classical calculus. Kolwankar [2] introduced the *local fractional derivative* (LFD) and proved that, in the singular points, the LFD is either zero or infinite; it assumes a finite value only if the order of derivation is equal to the Hölder exponent. For the devil's staircase graph of fig. 2a, for instance, the LFD of order equal to the dimension of the Cantor set is zero everywhere except in the singular points where it is finite. This leads to the definition of the fractal strain as the LFD of order $(1- d_\varepsilon)$ of the displacement. As inverse of the LFD, Kolwankar introduced the *fractal integral*. With this tool it is possible describing the mechanics of fractal media, calculating the size effects on tensile and flexural strength, and writing the equation of Principle of Virtual Work, which, in contrast with its continuum counterpart, has an intrinsic multi-scale structure.

References

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