

## NONLINEAR WAVE PROCESSES IN A BI-LAYER

Karima R. Khusnutdinova\*, Roger H. J. Grimshaw

*Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK*

e-mails: K.Khusnutdinova@lboro.ac.uk, R.H.J.Grimshaw@lboro.ac.uk

*Summary* Combinations of two materials are commonly used to obtain a structure with the properties better than that of the parent constituents. Coupled nonlinear Klein-Gordon equations can be used to study the properties of wave propagation in a bi-layer. Applying exact analytical methods, asymptotic analysis and numerical simulations we show that heterogeneity leads to a number of interesting nonlinear wave processes in bi-layers.

### MODEL OF A BI-LAYER

We study the one-dimensional nonlinear dynamics of a bi-layer using the model of coupled chains of particles [1], i.e. two one-dimensional periodic chains with linear links between elements and nonlinear interaction between the chains. This can be considered as a natural generalization of the Frenkel-Kontorova model [2]. The long-wave dynamics of this system is described in dimensionless variables by coupled nonlinear Klein-Gordon equations

$$u_{tt} - u_{xx} = f_u(u, w), \quad w_{tt} - c^2 w_{xx} = f_w(u, w), \quad (1)$$

where  $c$  is the ratio of characteristic speeds of non-interacting components. The function  $f(u, w)$  describes interaction between the chains, and should be found experimentally.

Similar two-component models were proposed to describe dynamics of hydrogen-bonded chains, molecular crystals and polymer chains, etc. (see review [3]), and also in connection with the crack propagation in solids (e.g., [4], [5], [6]). It's also worth noting that similar equations describe some processes in the DNA double helix (see [7] and the references there).

When the potential function is  $f(u, w) = \cos(\delta u - w) - 1$ , system (1) reduces to coupled sine-Gordon equations:

$$u_{tt} - u_{xx} = -\delta^2 \sin(u - w), \quad w_{tt} - c^2 w_{xx} = \sin(u - w), \quad (2)$$

where the variable  $u$  replaces  $\delta u$ , compared to system (1).

### NONLINEAR WAVE PROCESSES

The travelling wave solutions of (1) (in particular, solitary waves) can be found analytically. If the characteristic speeds of the non-interacting chains are different ( $c^2 \neq 1$ ), a gap appears in the velocity spectrum of the solitary waves, i.e. the system acts as a filter for solitary waves. Here the relative displacement ("upper" particles relative to the "lower" ones) remains the same as in the FK model (per period of the chain), but the absolute displacement depends on the velocity of the wave [1]. The influence of a delamination zone on kink propagation is studied numerically in [8].

Another essential feature of the considered system is the possibility of energy exchange between its physical components (layers) [9], in particular, in the situation when one component is initially excited (say,  $u$ ) with another component ( $w$ ) being initially at rest. Periodic and quasi-periodic processes in the wave system (2) are analogous to the energy exchange in a system of coupled pendulums in classical mechanics.

The energy exchange between the two components  $u$  and  $w$  of the system (2) takes place since different branches of the dispersion curve coexist for the same wavenumber  $k$ , which is typical for bi-layers.

We consider the structure and stability of solutions involving two waves or two pairs of counter-propagating waves and describing periodic energy exchange between the layers. The exact two-wave solution of system (2) for the periodic energy exchange is found for  $c = 1$  (Fig. 1). In the general case, it is possible to construct weakly nonlinear two- and four-wave solutions describing the energy exchange between two components  $u$  and  $w$ . Using both the theoretical predictions following from the asymptotic analysis and numerical simulations of the original unapproximated equations we show that these solutions may be modulationally unstable. These instabilities may lead to the formation of localized structures, and to a modification of the energy exchange between the components [10], [11] (Fig. 2).

---

\*On leave from: Institute of Mechanics, Ufa Branch of the Russian Academy of Sciences, Karl Marx Str. 6, Ufa, 450000, Russia

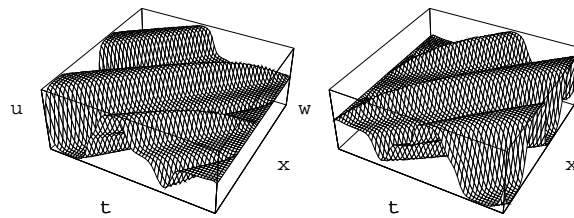


Figure 1: Energy exchange in exact two-wave solution.

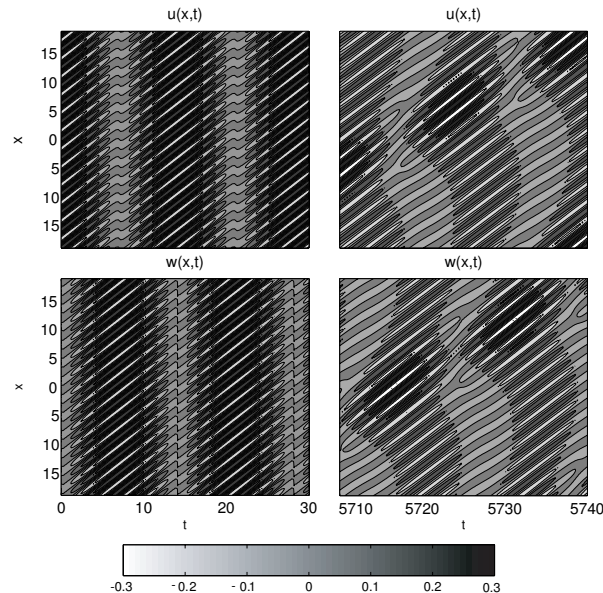


Figure 2: Energy exchange under the influence of modulational instability.

## CONCLUSIONS

A major difficulty in the application of lattice models to real physical or biological processes is in the derivation of the interaction potentials and parameters of the model. In the case of bi-layers, this should (and can) be done experimentally. Here, we consider a simple model situation close to the Frenkel-Kontorova model to study the peculiarities of nonlinear wave processes in bi-layers.

## References

- [1] Akhatov I.Sh., Baikov V.A., and Khusnutdinova K.R.: Non-linear dynamics of coupled chains of particles. *J. Appl. Maths Mechs* **59**: 353-361, 1995.
- [2] Kontorova T.A. and Frenkel Ya.I.: On the theory of plastic deformation and twinning I. *JETP* **8**: 89-95, 1340-1348, 1938.
- [3] Braun O.M. and Kivshar Yu.S.: Nonlinear dynamics of the Frenkel-Kontorova model. *Phys. Rep.* **306**: 1-108, 1998.
- [4] Slepyan L.I.: *Models and Phenomena in Fracture Mechanics*. Springer, Berlin 2002.
- [5] Smith E.: The effect of the discreteness of the atomic structure on cleavage crack extension: Use of a simple one-dimensional model.II. *Mater. Sci. & Engng* **30**: 15-22, 1977.
- [6] Ginzburg V.V. and Manevitch L.I.: The extended Frenkel-Kontorova model and its application to the problems of brittle fracture and adhesive failure. *Int. J. of Fracture* **64**: 93-99, 1993.
- [7] Yakushevich L.V.: *Nonlinear Physics of DNA*. Wiley, Chichester 1998.
- [8] Khusnutdinova K.R. and Silberschmidt V.V.: Lattice modelling of nonlinear waves in a bi-layer with delamination. *Proc. Estonian Acad. Sci. Phys. Math.* **52**: 63-75, 2003.
- [9] Khusnutdinova K.R. and Pelinovsky D.E.: On the exchange of energy in coupled Klein-Gordon equations. *Wave Motion* **38**: 1-10, 2003.
- [10] Griffiths S.D., Grimshaw R.H.J. and Khusnutdinova K.R.: The influence of modulational instability on energy exchange in coupled sine-Gordon equations, *Theor. Math. Phys.* **137**: 1446-1456, 2003.
- [11] Griffiths S.D., Grimshaw R.H.J. and Khusnutdinova K.R.: Nonlinear dynamics of two pairs of counter-propagating waves in two component media, *in preparation*.