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NONSTATIONARY FLOW OF STOKESIAN FLUID THROUGH ELASTIC SKELETON WITH HIERARCHICAL STRUCTURE

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Summary The aim of contribution is to consider the problem of nonstationary flow of v Stokesian fluid through linear elastic porous skeleton. The novelty consists in that the skeleton is deformable and characterized by the so-called double-porosity matrix. The material of the matrix is hierarchical with two *well-separated* scales ε and ε^2. Macroscopic equations were derived by using the method of multiscale convergence. Darcy's law involves both scales and is nonlocal in time.

INTRODUCTION

Porous media theories play an important role in many branches of engineering, including material science, the petroleum industry, chemical engineering and soil mechanics as well as in biomechanics. Powerful tool in the derivation of macroscopic equations describing flows through porous media are offered by homogenization method, cf. [2,5,7,9] and the references therein.

The aim of this contribution is to examine the dynamic flow of Stokesian fluid through elastic porous skeleton. The last is characterized by a hierarchical structure with two *well-separated* scales, cf. Allaire and Briane [1]. For the sake of simplicity these two scale are characterized by ε and ε^2, respectively. To find the macroscopic equations we use the (homogenization) method of three-scale asymptotic convergence, being a specific case of the method of multiscale convergence. To put it briefly, the macroscopic equations, including the Darcy law, are obtained by letting ε tend to zero. This law involves both scales and is nonlocal in time

MICROSCOPIC MODEL: PERIODIC DISTRIBUTION OF MICROPORES

Let Ω be an open domain in $\mathbb{R}^3$. We consider the open domain $\Omega_\varepsilon = \Omega \backslash S^\varepsilon_\Omega$. The closed set S^ε_Ω is obtained as follows: let Y and Z be two fixed reference cells $Y = [0, y^0_1] \times [0, y^0_2] \times [0, y^0_3]$, $Z = [0, z^0_1] \times [0, z^0_2] \times [0, z^0_3]$ and set $y^0 = (y^0_1, y^0_2, y^0_3)$, $z^0 = (z^0_1, z^0_2, z^0_3)$. Denote $F \subset Y$ and $S \subset Z$ two closed subset with smooth boundaries and nonempty interiors. We repeat F and S with "Y-periodicity" and "Z-periodicity", respectively. We assume that for any ε the period Y is exactly covered by a finite number of translated cells of $\frac{\varepsilon^2}{\varepsilon} Z$. The small parameter ε will tend to zero. We set $S^\varepsilon_Y = (Y \setminus F) \cap (\bigcup_{k \in K_\varepsilon} \frac{\varepsilon^2}{\varepsilon}(S + kz^0))$, $Y^\varepsilon = Y \setminus S^\varepsilon_Y$. We assume that $S^\varepsilon_Y \cap F = \emptyset$ for every $\varepsilon > 0$. Hence S^ε_Y is a subset of $Y \backslash F$, composed of closed sets periodically distributed with period $\frac{\varepsilon^2}{\varepsilon}$ and of the same size as the period. We set $S^\varepsilon_\Omega = \bigcup_{h \in H_\varepsilon} \varepsilon(S^\varepsilon_Y + hy^0)$, $F_\varepsilon = \Omega \cap (\bigcup_{k \in \mathbb{Z}^3} \varepsilon(F + ky^0))$, cf. Ene and Saint Jean Paulin [4]. The structure of Ω_ε presents a double periodicity (ε and ε^2). The zones in which the fractions are concentrated are ε-periodic and of size ε.

Let $\Omega^S_\varepsilon = \Omega \setminus \Omega_\varepsilon$ be a domain which is occupied by the elastic matrix and Ω_ε is occupied by the incompressible viscous fluid. The microscopic equations of the problem to be solved are as follows:

$$\varrho^S \ddot{\boldsymbol{u}}^\varepsilon(x,t) = \mathrm{div}\,[\boldsymbol{c} : \boldsymbol{e}(\boldsymbol{u}^\varepsilon(x,t))] + \boldsymbol{F}^S(x,t), \qquad (x,t) \in \Omega^S_\varepsilon \times (0,T), \tag{1}$$

$$\varrho^L \dot{\boldsymbol{v}}^\varepsilon(x,t) = \mu\varepsilon^2 \Delta \boldsymbol{v}^\varepsilon(x,t) - \nabla p^\varepsilon(x,t) + \boldsymbol{F}^L(x,t), \qquad (x,t) \in \Omega_\varepsilon \times (0,T), \tag{2}$$

$$\mathrm{div}\, \boldsymbol{v}^\varepsilon(x,t) = 0, \qquad (x,t) \in \Omega_\varepsilon \times (0,T), \tag{3}$$

$$\dot{\boldsymbol{u}}^\varepsilon(x,t) = \boldsymbol{v}^\varepsilon(x,t), \quad \text{on } \partial\Omega^S_\varepsilon \cap \partial\Omega_\varepsilon \times (0,T). \tag{4}$$

Boundary and initial conditions complete Eqs. (1)-(4). Here $\boldsymbol{e}(\boldsymbol{u}) = 1/2(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$.

MULTISCALE CONVERGENCE

To derive the macroscopic model and effective coefficients the so-called multiscale convergence method is used, see [1-6]. Double porosity requires exactly the three-scale convergence.

We say that the sequence of functions $\{u^\varepsilon\}$ three-scale converges to $u^0(x,y,z)$ iff

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \phi(x, x/\varepsilon, x/\varepsilon^2)\, dx = \int_\Omega \int_Y \int_Z u^0(x,y,z) \phi(x,y,z)\, dx dy dz, \text{ for each } \phi \in L^2(\Omega, Y \times Z).$$

MACROSCOPIC MODEL

As a first step in the process of homogenization we obtain the so-called *three-pressure system* being an extension of two-pressure system in the case of two-scale convergence method and only one porosity. After two step of averaging, first over the basic cell Z and then over Y, we obtain the macroscopic relations. The macroscopic model consists, among others, of the consolidation equation

$$\varrho^S\langle\ddot{\boldsymbol{u}}^0\rangle+\varrho^L\langle\dot{\boldsymbol{v}}^0\rangle=\operatorname{div}\boldsymbol{\sigma}-\nabla_x p^0-\int_{Y_L}\nabla p^1\,dy-\int_{Y_L}\int_{Z_L}\nabla_z p^2\,dzdy+(1-f)\boldsymbol{F}^S+f\boldsymbol{F}^L,\ \text{in }\Omega\times(0,T). \quad (5)$$

It can be shown that $p^0=p^0(x,t)$, $p^1=p^1(x,y,t)$ and $p^2=p^2(x,y,z,t)$. The three-pressure system is as follows in $\Omega\times Y_L\times Z_L\times(0,T)$

$$\varrho^L\dot{\boldsymbol{v}}^0_{rel}(x,y,z,t)=\mu\Delta_z\boldsymbol{v}^0_{rel}+\boldsymbol{F}^L(x,t)-\varrho^S\ddot{\boldsymbol{u}}^0(x,t)-\nabla_x p^0(x,t)-\nabla_y p^1(x,y,t)-\nabla_z p^2(x,y,z,t), \quad (6)$$

where $\boldsymbol{v}_{rel}(x,y,z,t)=\boldsymbol{v}^0(x,y,z,t)-\dot{\boldsymbol{u}}^0(x,t)$ and f means the porosity. Moreover we obtain

$$\operatorname{div}_z\boldsymbol{v}^0_{rel}(x,y,z,t)=0\quad\text{in }\Omega\times Y_L\times Z_L\times(0,T),$$

$$\operatorname{div}_y\int_{Z_L}\boldsymbol{v}^0_{rel}(x,y,z,t)\,dz=\frac{1}{|Z|}\int_{\Gamma_Z}\dot{\boldsymbol{u}}^1(x,y,z,t)\cdot\boldsymbol{N}\,ds(z)\quad\text{in }\Omega\times Y_L\times(0,T),$$

$$\operatorname{div}_x\int_{Y_L}\int_{Z_L}\boldsymbol{v}^0_{rel}(x,y,z,t)\,dzdy=\frac{1}{|Y|}\int_{\Gamma_Y}\int_Z\dot{\boldsymbol{u}}^1(x,y,z,t)\boldsymbol{n}\,dz\,ds(y)+\int_Y\int_Z\dot{\boldsymbol{u}}^2(x,y,z,t)\cdot\boldsymbol{N}\,dyds(z)\ \text{in }\Omega\times(0,T).$$

Here $\boldsymbol{N}$ is the unit normal vector to the boundary of Z_L ($\partial Z_L=\Gamma_Z$) while $\boldsymbol{n}$ is the unit normal vector to ∂Y_L. The three-pressure system is a combination of usual homogenization and cell problem. By eliminating the microscopic variables z and y from Eq. (6) the homogenized Darcy law nonlocal in time is obtained. First, we separate the variables: $p^2(x,y,z,t)=\Pi^{(m)}(z,t)[\boldsymbol{F}^L(x,t)-\varrho^L\ddot{\boldsymbol{u}}^0(x,t)-\frac{\partial p^0(x,t)}{\partial x_m}-\frac{\partial p^1}{\partial y_m}(x,y,t)]$. Then we get

$$\langle v^0_{m\ rel}\rangle_{Z_L}=\frac{1}{\varrho^L}\int_0^t\langle\chi^{(s)}_m(z,t-\tau)\rangle_{Z_L}[F_s-\varrho^L\ddot{u}_s-\frac{\partial p^0}{\partial x_s}-\frac{\partial p^1(x,y,t)}{\partial y_s}]\,d\tau,\quad\text{in }\Omega\times Y_L\times(0,T) \quad (7)$$

where functions $\Pi^{(m)}(z,t)$ and $\chi^{(s)}_m(z,t)$ satisfy the so-called local cell problem posed on $Z_L\times(0,T)$.

$$\mu\Delta_{zz}\chi^{(s)}_m=\varrho^L\dot{\chi}^{(s)}_m-(e^{(s)}_m-\frac{\partial\Pi^{(s)}}{\partial z_m}),\qquad\operatorname{div}_z\boldsymbol{\chi}^{(s)}(z,t)=0\quad\text{in }Z_L\times(0,T).$$

Here $\boldsymbol{e}^{(s)}$, $s=1,2,3$ stands for the vector basis in $\mathbb{R}^3$. We observe that in Eq. (7) the local variable y is present. To obtain fully homogenized Darcy law it is sufficient to eliminate y by integration of both sides of Eq. (7) over the cell Y.

To pass to the stationary flow we have to pass with time to infinity ($t\to\infty$).

FINAL REMARKS

To derive macroscopic equations in the case of randomly distributed pores in hierarchical porous deformable medium one can use the method of stochastic multiscale convergence in the mean, cf. [8]. The method of three-scale convergence used in this contribution can be extended to more complex hierarchical media with mixed periodic-random distribution of pores. The approach to macroscopic modelling proposed by us [8] applies not only to geomaterials like fractured porous media, but also to biological tissues (for instance bone tissue).

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