NONLINEAR OSCILLATORY CONVECTION IN MUSHY LAYERS

Peter Guba, M. Grae Worster
Institute of Theoretical Geophysics, DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

Summary The nonlinear development of two-dimensional oscillatory convection in a mushy layer during solidification of a binary mixture is studied. A comprehensive investigation of the nonlinear stability of travelling and standing waves, including the relative stability between the two wave patterns, is provided. The relevance of our results to recent laboratory experiments is discussed.

INTRODUCTION

When a binary alloy solidifies directionally, a planar solidification front can become morphologically unstable owing to compositional supercooling. Consequently, regions of coexisting liquid and solid phases, referred to as 'mushy' regions, are often formed. The solidification of a binary alloy in a mushy layer can be profoundly influenced by fluid flow. The flow in mushy layers can be driven by a variety of different mechanisms, though buoyancy-driven convection has received most attention in recent years [1]. As a result of a complex interaction with solidification, convection can be oscillatory at onset [2]. It is the nonlinear evolution of this oscillatory instability which we shall elucidate here.

FORMULATION

The physical system under consideration consists of a horizontal mushy layer lying between semi-infinite solid and liquid regions. The system is cooled from below with the rate of freezing being constant in time, so that the solid–mush and mush–liquid interfaces advance upward with a constant solidification speed. The dimensionless equations for the temperature (or equivalently composition), the local solid fraction and the Darcy fluid velocity assume the form given in [3]. Convective instability is controlled principally by the Rayleigh number R, mediated by the Stefan number S and the compositional ratio $\mathcal{C}$.

In order to pose an analytically tractable problem, we suppose that the system is close to the eutectic point [4], which amounts to considering the limit of small mushy-layer thickness compared to the thermal-diffusion lengthscale $\delta \ll 1$, with $\mathcal{C} = O(1/\delta)$. We assume also that the Stefan number is large, $S = O(1/\delta)$, which allows a destabilization of the system to the oscillatory mode of convection [2]. In this limit, a function relating the permeability of the mushy layer to the local solid fraction can be approximated by $K(\phi) \sim 1 + K_1\phi + K_2\phi^2$, where K_1 and K_2 are positive constants.

BIFURCATION ANALYSIS

Oscillatory instability

From a linear stability analysis of the steady, motionless basic state [5], it is known that the onset of convection in the mushy layer can be oscillatory, even though the stabilizing thermal buoyancy is set equal to zero and no double-diffusive effects are present in the model. In physical terms, this oscillatory instability is linked with a phase lag between the background macroscopic solidification and the local dissolution caused by the fluid motion.

Weakly-nonlinear oscillatory convection

A key result of our analysis is a set of coupled evolution equations

$$a\dot{\mathcal{A}} = b\mathcal{A} + c\mathcal{A}|\mathcal{A}|^2 + d\mathcal{A}|\mathcal{B}|^2, \tag{1}$$

$$a\dot{\mathcal{B}} = b\mathcal{B} + c\mathcal{B}|\mathcal{B}|^2 + d\mathcal{B}|\mathcal{A}|^2, \tag{2}$$

for the complex amplitudes $\mathcal{A}$ and $\mathcal{B}$ of left- and right-travelling waves, respectively. These equations describe the nonlinear interaction between the two counter-propagating waves near a Hopf bifurcation point [6] and can be used to determine the stability of, and the relative stability between, the travelling and standing waves.

The structure of the bifurcation diagrams in the vicinity of the linear Rayleigh number for the onset of oscillatory convection $R = R^{(o)}$ is illustrated in figure 1. Both travelling and standing waves bifurcate simultaneously. Stable solutions are present provided both oscillatory branches bifurcate supercritically. Furthermore, the stable solution is the one with the larger Nusselt number ($\sim 1 + \epsilon^2(|\mathcal{A}|^2 + |\mathcal{B}|^2)$, where ϵ measures a perturbation amplitude).

The explicit predictions for the mushy-layer system can be established on the basis of the complex coefficients a–d. The presence of oscillations is controlled by the parameters S, $\mathcal{C}$ and δ, as can be inferred from [5]. Our analysis reveals that a preference for distinct wave patterns, and hence the possibility of their realization in the experiments, is determined by particular values of the permeability coefficients K_1 and K_2, as indicated in figure 2. Thus, the pattern selection process is dominated by a nonlinear effect associated with permeability variations due to perturbations to the solid fraction.

Recently, Solomon & Hartley [7] performed experiments on an aqueous solution of ammonium chloride, and were able to observe a periodic convective behaviour. However, the sense of the convective motion did not reverse during evolution, indicating that the observed behaviour should not be related to the standing roll state studied here. Instead, we anticipate that it is a manifestation of a secondary Hopf bifurcation, arising from an intricate nonlinear interaction between steady and oscillatory convection. The present study forms the basis for future investigation of this problem.

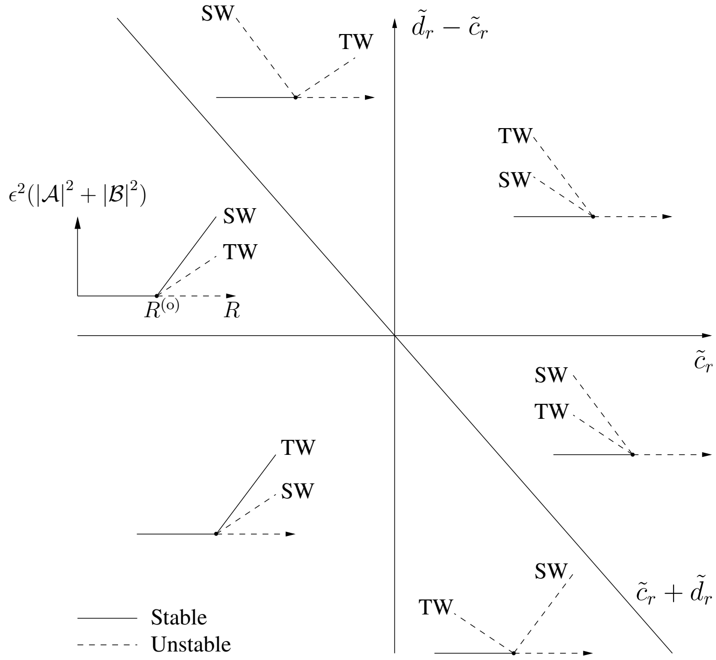

Figure 1. Bifurcation diagrams in the $(\tilde{c}_r, \tilde{d}_r - \tilde{c}_r)$ plane, showing the amplitude $\epsilon^2(|\mathcal{A}|^2 + |\mathcal{B}|^2)$ of travelling waves (TW) and standing waves (SW) as a function of the Rayleigh number R. Here, $\tilde{c}_r = (c/a)_r$, $\tilde{d}_r = (d/a)_r$, and the subscript r indicates the real part.

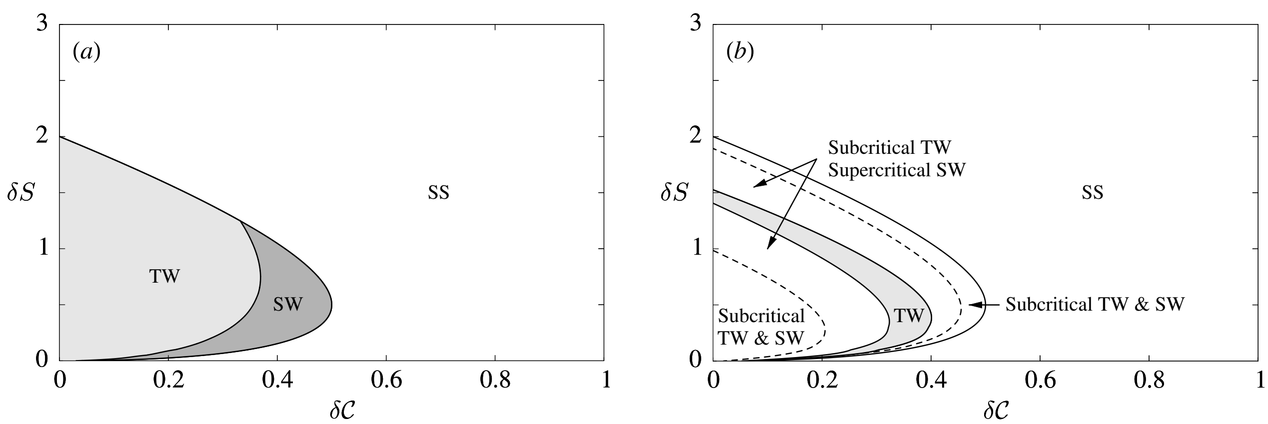

Figure 2. Stability regions. A diagram showing the regions in the $(\delta S, \delta \mathcal{C})$ parameter space where various features of oscillatory convection can be identified. The light-shaded portion corresponds to those parameter values where stable travelling waves are predicted, while the dark-shaded portions indicate where stable standing waves are expected to appear. The outermost solid curve represents the boundary between steady (SS) and oscillatory behaviour. The dashed lines separate the domains in which either oscillatory bifurcation is subcritical. Two representative cases are shown: (*a*) $K_1 = 0$, $K_2 = 0.05$; (*b*) $K_1 = 3$, $K_2 = 6$.

CONCLUSIONS

We have studied the nonlinear development of the oscillatory convecting states in mushy layers. Depending on parameter values, either travelling waves or standing waves can be stable at onset. Our analysis emphasizes the importance of a specific form of the constitutive relationship in selecting the oscillatory convection pattern. The overstable convective motions in the mushy layer leave a signature in the solid in the form of striation of its texture and local bulk composition. Quantification of these issues, currently under way, promises to provide a useful insight into the nonlinear aspects of segregation in cast alloys.

References

[1] Worster M. G., *Ann. Rev. Fluid Mech.* **29**: 91–122, 1997.

[2] Anderson, D. M., Worster, M. G., *J. Fluid Mech.* **302**: 307–331, 1995.

[3] Amberg, G., Homsy, G. M., *J. Fluid Mech.* **252**: 79–98, 1993.

[4] Fowler, A. C., *IMA J. Appl. Maths* **35**: 159–174, 1985.

[5] Anderson, D. M., Worster, M. G., *J. Fluid Mech.* **307**: 245–267, 1996.

[6] Crawford, J. D., Knobloch, E., *Ann. Rev. Fluid Mech.* **23**: 341–382, 1991.

[7] Solomon, T. H., Hartley, R. R., *J. Fluid Mech.* **358**: 87–106, 1998.