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ON CONVEXITY CONDITIONS IN SPATIAL AND MATERIAL SETTINGS OF HYPERELASTICITY

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Summary In this contribution convexity conditions for the spatial and material motion problem are investigated. Whereas the spatial motion problem corresponds to the usual equilibrium equations, the material motion problem is driven by the inverse deformation gradient, thus it deals with material or configurational forces that are energetically conjugated to material variations, i.e. variations at fixed spatial positions.
The duality between the two problems is elaborated in terms of balance laws, their linearisations including the consistent tangent operators and in particular the so-called acoustic tensors.
Issues of convexity and in particular of rank-one-convexity are discussed in both settings. As a remarkable result it turns out, that if the rank-one-convexity condition is violated, the loss of well-posedness of the governing equations occurs simultaneously in the spatial and in the material motion problem.
Thus, the inclusion of the material motion problem does not lead to additional requirements to maintain rank-one-convexity or ellipticity.
The results are developed for the hyperelastic case in general and highlighted analytically and numerically for a material of Neo-Hookean type.

EXTENDED ABSTRACT

The motion of a continuous medium can be described via a Lagrangian or an Eulerian parametrisation. In the former, material particles are followed in space, whereas in the latter, material flow along fixed spatial positions is tracked. This is closely related to spatial and material settings of continuum mechanics, see e.g. [1, 3, 5]. In the spatial setting, spatial or physical forces are dealt with that are energetically conjugated to spatial variations of fixed material points. This is the viewpoint of classical mechanics involving forces in the sense of Newton. Conversely, in the material setting, material or configurational forces are encountered that are energetically conjugated to material variations at fixed spatial positions. This is the viewpoint of configurational mechanics involving forces in the sense of Eshelby.

The material setting of continuum mechanics has received considerable attention in recent years due to its theoretical as well as its numerical implications, see e.g. [2, 6, 8, 11, 13], whereby in particular the duality between the spatial and material setting has been elucidated for various cases and applications, see e.g. [4, 7, 12]. The related contributions of this group [1-13], which contain also the relevant references, are listed in the attached bibliography.

Material force residuals arise as a result of inhomogeneities and/or heterogeneities. Thus, the study of the material motion problem is relevant for problems involving propagating defects e.g. such as cracks, interfaces, voids and inclusions.

However, (spurious) material force residuals may also appear as a consequence of non-optimal finite element discretisations, which can be considered as numerical or algorithmic (as opposed to physical) inhomogeneities. Indeed, by moving nodes opposite to the (spurious) material force residual additional energy is released and vanishing (spurious) material force residuals can be obtained.

This has inspired the idea to consider in addition to the spatial motion problem the material motion problem whereby both the spatial coordinates and the material coordinates are taken as simultaneous degrees of freedom. Thus, an Arbitrary Lagrangian-Eulerian (ALE) parametrisation is obtained in a natural manner, see [9, 10]. It has been shown in [9, 10] that the meshes obtained via this dual equilibrium problem are optimal in the sense that the potential energy is the lowest possible for the given element connectivity. Thus the material motion problem can be used as a mesh optimisation method.

A relevant issue in the above methodology is the analysis of convexity of the spatial and the material motion problem. In the infinitesimal context, the convexity condition is strongly related to the classical criterion of uniqueness in terms of the positive definiteness of the second order work. The general convexity condition, however, is typically too restrictive from a physical point of view and, moreover, too difficult to evaluate from a mathematical point of view.

A weaker condition, yet having a well-established mathematical status is the condition of rank-one-convexity. In the infinitesimal sense, rank-one-convexity corresponds to the classical Legendre-Hadamard condition which dates back to the early work of Hadamard on the propagation of waves. For the class of non-transient problems considered herein, the loss of rank-one-convexity is one-to-one related to the loss of ellipticity which is typically accompanied by the loss of well-posedness of the governing equations. The elaboration of the rank-one-convexity condition is thus of fundamental relevance, not only for the spatial but also for the material motion problem.

Although the notions of convexity, uniqueness, monotony, stability, rank-one-convexity, ellipticity and well-posedness are well-established in the spatial motion context, the appropriate classification of the material motion problem turns out to be rather cumbersome. Nevertheless, in particular a difference between the two settings in this respect would be significant, for instance if the material motion problem would lose rank-one-convexity or rather ellipticity at an earlier stage of loading than the spatial motion problem, or vice versa.

After reiterating the governing equations in both settings, the convexity conditions of the two problems are discussed. To this end, relevant tangent operators are derived through a consistent linearisation of the corresponding problem. Next we elaborate the issue of rank-one-convexity which can be expressed in terms of the acoustic tensors. It turns out that their relation is of remarkable simplicity: the acoustic tensor of the one setting can be obtained from the acoustic tensor of the other setting by a double push forward / pull back and a scaling in terms of the applied stretch!

As a consequence, a loss of ellipticity occurs simultaneously in both settings. This implies that the material motion problem offers no additional difficulties compared to the spatial motion problem, as regards the loss of well posedness.

For illustration purposes, the corresponding analyses are exemplified for a Neo-Hookean material, whereby the eigenvalues of both acoustic tensors are investigated.

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