STANDING GRAVITY WAVES IN DEEP WATER

Gérard Iooss*, Pavel Plotnikov†, John F.Toland‡,
* *IUF, INLN UMR CNRS-UNSA 6618, 1361, route des Lucioles, 06560 Valbonne, France*
† *Lavryentyev Inst. of Hydrodynamics, Lavryentyev pr. 15, Novosibirsk 630090, Russia*
‡ *Dept. of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*

Summary The two dimensional standing gravity wave problem, for an infinitely deep layer, is considered, based on the formulation of the problem as a second order non local PDE (see (1)). Despite the infinitely many resonances in the linearized problem we use the Nash-Moser implicit function theorem for proving the existence of standing waves corresponding to values of the amplitude ε having 0 as a Lebesgue point.

INTRODUCTION

We consider an infinitely deep layer of a perfect incompressible fluid, in two-dimensional potential motion under gravity, with a free surface without surface tension. We are interested in *solutions which are periodic in time and in the horizontal direction, and invariant under reflexion in the vertical axis.* The existence of solutions in the finite depth case were proved recently by Plotnikov and Toland in [5]. The problem we consider below has the additional difficulty to be infinitely degenerate at the linearized level, as noticed a long time ago (Poisson (1816), Boussinesq (1877), Rayleigh (1915)).

The formulation we take for the standing wave problem is the one introduced in Dyachenko et al [2], where w is a function of (x,t), both even and $2\pi-$ periodic in x and t. In the following we denote by $\mathcal{H}$ the spatial periodic Hilbert transform. The equation satisfied by w is a *second order nonlocal partial differential equation*, which may be written as follows (see [2] and [4])

$$\mathcal{F}(w,\mu) \stackrel{def}{=} \partial_t(L_{w'}\dot{w}) - \mu\mathcal{H}w' + \mathcal{H}\partial_x\left\{\frac{1}{D}\mathcal{H}(L_{w'}\dot{w}\mathcal{H}L_{w'}\dot{w}) + (\mathcal{H}L_{w'}\dot{w})\mathcal{H}\left(\frac{1}{D}L_{w'}\dot{w}\right)\right\} = 0, \qquad (1)$$

where a dot means a partial time (t) derivative, and a prime means a partial space (x) derivative ∂_x, and where

$$L_{w'}f = (1+\mathcal{H}w')f - w'\mathcal{H}f, \;\; D = (1+\mathcal{H}w')^2 + w'^2,$$

and μ is the bifurcation parameter $\mu = gT^2/2\pi\lambda$, where T is the time period, λ is the spatial period, and g is the gravitational acceleration. With these equations, the free surface is given parametrically in physical coordinates (ξ,η) by

$$(\xi,\eta) = (x+\mathcal{H}w(x,t), -w(x,t)); \;\; (x,t)\in\mathbb{R}^2.$$

The formulation (1) is equivalent to the classical formulation for sufficiently smooth solutions. We notice that $w = const$ is always a solution, and that we can look for solutions of (1) with zero average. Let us define for $m \geq 0$ suitable (Sobolev) spaces of doubly periodic functions

$$H_{\natural\natural}^{m,ee} = \{w \in H^m\{(\mathbb{R}/2\pi\mathbb{Z})^2\};\; w \text{ is even in } x \text{ and in } t,\; w \text{ has zero average}\},$$

Then, for $m \geq 3$, $\mathcal{F}$ is an analytic map from $H_{\natural\natural}^{m,ee} \times \mathbb{R}$ to $H_{\natural\natural}^{m-2,ee}$ and the linearization of (1) at the origin reads

$$\mathcal{L}_\mu u \stackrel{def}{=} D_w\mathcal{F}(0,\mu)u = \ddot{u} - \mu\mathcal{H}\partial_x u.$$

If μ is irrational, the kernel of $\mathcal{L}_\mu$ in $H_{\natural\natural}^{m,ee}$ is $\{0\}$. For $\mu = 1$ the kernel of $\mathcal{L}_1$ in $H_{\natural\natural}^{m,ee}$ is the *infinite-dimensional subspace* $E_0 \cap H_{\natural\natural}^{m,ee}$, with $E_0 = span\{A_q \cos q^2x \cos qt; A_q \in \mathbb{R},\;\; q \in \mathbb{N}\}$. For other rational values of μ, the infinite dimensional kernel of $\mathcal{L}_\mu$ is easily deduced from E_0. For $f \in H_{\natural\natural}^{m,ee}$ which is orthogonal to E_0 in $L_{\natural\natural}^2$, there is a unique $u \in H_{\natural\natural}^{m,ee}$, orthogonal to E_0 in $L_{\natural\natural}^2$, which is solution of $\mathcal{L}_1 u = f$.

Since the pseudo-inverse of $\mathcal{L}_1$ is not regularizing and since the nonlinear terms in (1) lose 2 degrees of regularity, we need to adapt the Nash-Moser implicit function theorem for the bifurcation problem.

MAIN RESULTS

A suitable choice of scales allows to only consider μ close to 1. It is then known that approximate solutions of the standing wave problem exist up to an arbitrary power of ε, where $\mu = 1 + \varepsilon^2/4$. More precisely we have the following

Theorem 1 *There are infinitely many approximate solutions w of (1) under the form of asymptotic expansions in powers of the amplitude ε of the wave*

$$w_\varepsilon^{(N)} = \sum_{1\leq n\leq N} \varepsilon^n w^{(n)},\ \mu = 1+\varepsilon^2/4,\ w^{(1)} = \sum_{q\in I} \frac{\varepsilon_q}{q^2}\cos q^2 x \cos qt,\ \varepsilon_q = \pm 1,\ I\subset \mathbb{N},$$

with any finite subset I of $\mathbb{N}$, moreover $w_\varepsilon^{(N)}$ is such that $\mathcal{F}(w_\varepsilon^{(N)}, 1+\varepsilon^2/4) = \varepsilon^{N+1}Q_\varepsilon$, where Q_ε is bounded in any $H_{\natural\natural}^{m,ee}$ when $\varepsilon \to 0$ (see [1] for $I=\{1\}$, [3] for the general result).

Our main result is the following

Theorem 2 *For $m\geq 17$, $N\geq 4$ and $\varepsilon_0>0$ small enough, there is a measurable set $\mathcal{E}\subset(0,\varepsilon_0)$, with 0 as a Lebesgue point (hence dense at 0), i.e.*

$$(1/r)meas\{\mathcal{E}\cap(0,r)\} \to 1 \text{ as } r\to 0, \tag{2}$$

such that for any $\varepsilon\in\mathcal{E}$ and $\mu = 1+\varepsilon^2/4$, there exists a solution w of $\mathcal{F}(w,\mu)=0$ in $H_{\natural\natural}^{m,ee}$, whose asymptotic expansion is $w_\varepsilon^{(N)}$, with $w^{(1)} = \cos x\cos t$ and $||w - w_\varepsilon^{(N)}||_{H^m} = O(\varepsilon^{N-1})$ (see the complete proof in [6]).

Remark 1: Same theorem holds when we take any rational value of μ, instead of 1.

Remark 2: Theorem 2 also holds for most of the other type of asymptotic form $w_\varepsilon^{(N)}$ of the solution w, indicated at theorem 1, provided that a generally satisfied algebraic condition holds on the subset I of integers. Notice that in this case the set $\mathcal{E}$ of "good values" of ε may depend on the form of $w^{(1)}$ (see [7]).

SKETCH OF PROOF

Since there are infinitely many formal solutions, we need to specialize the solution we are looking for in searching for $\underline{w}$ of order 1, defined by

$$w = w_\varepsilon^{(N)} + \varepsilon^{N-1}\underline{w},\ N\geq 4,\ \mu = 1+\varepsilon^2/4 \tag{3}$$

and the equation to solve is now

$$\mathcal{F}(w_\varepsilon^{(N)} + \varepsilon^{N-1}\underline{w}, 1+\varepsilon^2/4) = 0. \tag{4}$$

The proof of Theorem 2 relies on the possibility to invert the linear operator $\mathcal{A}_w$ (approximate linearized operator at a non zero point), defined for any u in $H_{\natural\natural}^{s,ee}$ by

$$\partial_w \mathcal{F}(w, 1+\varepsilon^2/4)u = \mathcal{A}_w u + \Gamma(\mathcal{F}, u)$$

where $\Gamma(\mathcal{F},u)$ cancels when w is a solution of $\mathcal{F}(w,1+\varepsilon^2/4)=0$. The main result is that we are able to invert $\mathcal{A}_w$ for $\underline{w}$ bounded in $H_{\natural\natural}^{m,ee}$, $m\geq 17$, in controling a *diophantine condition on* two functions of ε arising after averaging, along the Newton iteration scheme used in the Nash-Moser theorem. This inverse operator satisfies

$$||\mathcal{A}_w^{-1}|| \leq c/\varepsilon^2 \tag{5}$$

in the space $\mathcal{L}(H_{\natural\natural}^{s,ee}, H_{\natural\natural}^{s-2,ee})$, $s\geq 2$, which shows a loss of two derivatives and a bound which grows like ε^{-2} as $\varepsilon\to 0$. For obtaining such a result, suitable for proving theorem 2, we manage a succession of change of coordinates and change of variables in the spirit of the work [5], except the first change of coordinates which reduces all second order derivatives occuring in $\mathcal{A}_w$ to a single second order derivative in t. The other key point is that we are able to invert explicitely the main order ε^2 of the "bifurcation part" of the linear equation $\mathcal{A}_w u = f$ in the infinite dimensional subspace $E_0\cap H_{\natural\natural}^{m,ee}$, which introduces the factor $1/\varepsilon^2$ in the bound (5) of $\mathcal{A}_w^{-1}$, while the control of the diophantine condition leads to the loss of the two derivatives.

References

[1] C.J.Amick, J.F.Toland. The semi-analytic theory of standing waves. Proc. Roy. Soc. Lond. A 411 (1987), 123-138.
[2] A.I.Dyachenko, E.A.Kuznetsov, M.D.Spector, V.E.Zakharov. Physics Lett. A **221** (1996), 73-79.
[3] G.Iooss. On the standing wave problem in deep water. J.Math. Fluid Mech. 4 (2002), 155-185.
[4] G.Iooss, J.F.Toland. Riemann-Hilbert and variational structure for standing waves. Far East J. Appl. Math., 15 (2004) 3, 345-376.
[5] P.I.Plotnikov, J.F.Toland. Nash-Moser theory for standing water waves. Arch. Rat. Mech. Anal. **159**, (2001) 1-83.
[6] G.Iooss, P.I.Plotnikov, J.F.Toland. Standing waves on an infinitely deep perfect fluid under gravity. Preprint INLN, 03/ 2004 (112 p.).
[7] G.Iooss, P.I.Plotnikov. Existence of multimodal standing gravity waves. Preprint 07/2004.