MODIFIED SHALLOW WATER EQUATION FOR INVISCID GRAVITY CURRENTS

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Summary We derive a modified shallow water equation for inviscid gravity currents in which the resistance from ambient fluid is taken into account. The resultant equations are highly nonlinear, which can be solved by a similarity transformation when the gravity current moving with a constant speed is considered. The similarity solutions are shown to be exactly the same with previous solutions obtained by traditional approach that the tradition shallow water equation is solved with the front condition. This result confirms the properness of the modified shallow water equation that it can govern the motion of inviscid gravity current with a resistance imposed by the ambient fluid.

DERIVATION OF MODIFIED SHALLOW WATER EQUATIONS

To analyze the motion of gravity currents, a common approach was to solve the hyperbolic shallow water equations together with the boundary conditions at both the current source at upstream and the current front at downstream [1]. The use of the front condition is a “logical reconciliation” [2] because this condition accounts for the force balance between the static pressure of the current and the resistance from the ambient fluid, which, nevertheless, is missed in the shallow water equations. In the present study, we start from the continuity and inviscid momentum equation and apply the shallow water approximation to derive the so-called modified shallow water equations, in which the resistance from ambient fluid is taken into account so that the consideration of the front condition becomes non-necessary. After applying both the kinematic and kinetic boundary conditions on the interface between the gravity current and the ambient fluid, we obtain the continuity and modified shallow water equations as follows

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}=0\,,\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g_r\frac{\partial h}{\partial x}=\frac{\gamma}{2}\frac{\partial}{\partial x}\left[\frac{(\partial uh/\partial x)^2}{1+(\partial h/\partial x)^2}\right]+\gamma\frac{\partial}{\partial t}\left(\frac{(\partial uh/\partial x)(\partial h/\partial x)}{1+(\partial h/\partial x)^2}\right) \tag{1,2}$$

where $u=u(x,t)$ is the flow velocity, $h=h(x,t)$ is the height of the current, $\gamma=\rho_2/\rho_1$, $g_r=(1-\gamma)g$ is the reduced gravitational acceleration, and ρ_1 and ρ_2 are densities of the current and the ambient fluid, respecitvely. It is noted that the right hand side of Eq. (2) vanishes when $\partial h/\partial x\approx 0$, implying that it accounts for the resistance force in terms of a form drag imposed by the ambient fluid on the current head where $\partial h/\partial x>0$ This term vanishes also for $\gamma\to 0$, or the non-Boussinesq gravity current whose density is much larger than the ambient fluid is considered.

THE SIMILARITY SOLUTION FOR A SPECIAL CASE

Equations (1) and (2) are highly nonlinear partial differential equations, being very difficult to solve even numerically. But, fortunately, for a special case that the gravity current moving with a constant speed and the shape of the gravity current remains similar all the time, they can be transformed by the similarity transformation, $\eta=x-Ut$, $u=U\tilde{u}(\eta)$ and $h=H\tilde{h}(\eta)$ into the following ordinary differential equations

$$-\frac{d\tilde{h}}{d\eta}+\frac{d\tilde{u}\tilde{h}}{d\eta}=0\,,\quad -\frac{d\tilde{u}}{d\eta}+\tilde{u}\frac{d\tilde{u}}{d\eta}+\frac{g_rH}{U^2}\frac{d\tilde{h}}{d\eta}=-\frac{\gamma}{2}\frac{d}{d\eta}\{(H\frac{d\tilde{h}}{d\eta})^2/[1+(H\frac{d\tilde{h}}{d\eta})^2\}\,. \tag{3,4}$$

We integrate Eqs. (3) and (4) once and apply relevant boundary conditions to determine the integral constants, and end up with the implicit relation of the current profile $\tilde{h}$ as follows

$$\frac{\eta}{H} = (\tilde{h}_f - \tilde{h})\sqrt{\frac{\gamma}{2(\tilde{h}_f - \tilde{h})} - 1} - \frac{\gamma}{4}\tan^{-1}\left[1 + \frac{\gamma}{4(\tilde{h}_f - \tilde{h}) - 2\gamma}\right]\sqrt{\frac{\gamma}{2(\tilde{h}_f - \tilde{h})} - 1} \qquad (5)$$

where h_f is the current height far away from the front. A numerical solution of Eq. (6) for $\gamma = 1.1$ is shown in Fig. 1, in which one can see the current profile is more like those observed in experiments than previous similarity solutions obtained by solving the tradition shallow water equation [1,2].

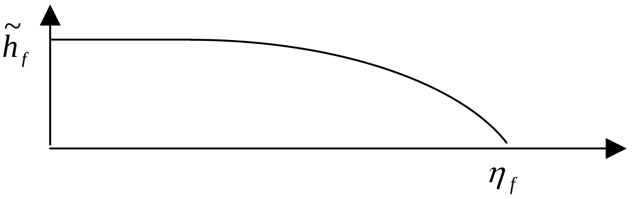

Fig. 1 A example of the current profile of present solution

To compare with previous results quantitatively, we employed the front condition $U^2 = \beta^2 g_r h_f$ [3] to rearrange Eq. (5) further and end up with the following simplified equation

$$H\frac{d\tilde{h}}{d\eta} = -\sqrt{\frac{2 - 2\beta^2\tilde{h}}{2\beta^2\tilde{h} - 2 + \beta^2\gamma}}\ . \qquad (6)$$

Due to the fact that the current height is zero at the foremost point of the current head and the slope of the current profile near the head is highly steep, namely, $d\tilde{h}/d\eta \to \infty$, we obtain

$$\beta = \sqrt{2/\gamma}\ , \qquad (7)$$

which is exactly the same with the formula provided by Simpson and Britter [3].

CONCLUSIONS

We have derived a modified shallow water equation in which the resistance in terms of the form drag imposed by the ambient fluid on the current head is included. The resultant equations are highly nonlinear, which, for the gravity current moving with a constant speed, can be solved by a similarity transformation, and the similarity solution is shown to be exactly the same with the formula derived by Simpson and Britter [3]. This comparison supports that the present modified shallow water equation can govern the motion of the inviscid gravity currents resisted by the form drag imposed by ambient fluid.

References

[1] Benjamin, T. B.: Gravity currents and related phenomena, *J. Fluid Mech.* **31**, 209, 1968

[2] Klemp, J. B., Rotunno, R., Skamarock, W. C.: On the dynamics of gravity currents in a channel, J. Fluid Mech. **269**, 169, 1994.

[3] Simpson, J. E., Britter, R. E.: The dynamics of the head of a gravity current advancing over a horizontal surface, J. Fluid Mech. **94**, 477, 1979.