REDUCTION OF MULTIDIMENSIONAL FLOW TO LOW DIMENSIONAL MAP FOR PIECEWISE SMOOTH SYSTEM EXPERIENCING CHAOS

Ekaterina E. Pavlovskaia and Marian Wiercigroch
Centre for Applied Dynamics Research School of Engineering and Physical Sciences, University of Aberdeen King's College, Aberdeen AB24 3UE, UK

Summary We consider dynamics of the piecewise smooth nonlinear systems for which general methodology of reducing multidimensional flows to low dimensional maps is proposed. This includes creation of the global map by stitching together local maps, which are constructed in the smooth sub-regions of phase space. Full details are given for a case study of drifting impact oscillator where five-dimensional flow is reduced to one dimensional (1D) approximate map. An appropriate co-ordinate transformation allowed the drift to be de-coupled from the bounded system oscillations. For these oscillations an exact two-dimensional map has been formulated and analysed. A further reduction to 1D approximate map is possible and will be discussed in the lecture. A standard nonlinear dynamic analysis reveals a complex behaviour ranging from periodic oscillations to chaos, and co-existence of multiple attractors. Accuracy of the constructed maps by comparing the dynamics responses with the exact solutions for a wide range of system parameters will be examined.

MULTIDIMENSIONAL FLOWS

Dynamics of vast majority of physical systems can be defined as multi-dimensional flows. If these flows are described by linear differential equations there are well developed mathematical theories, which can provide analytical solutions. However, a good deal of multi-dimensional flows is governed by nonlinear differential equations and, in particular, piecewise smooth differential equations, which naturally brings complications in providing effective solutions. For example the engineering systems with online control require simple robust models. One way of obtaining them is reduction of differential flows to iterative maps obviously if such transformation is feasible. In the published literature there is practically no systematic examples how such maps are built, therefore, in this article a sketch of such process is provided for an impact oscillator with drift.

We consider a drifting oscillator [1], where a mass is driven by an external force containing static and dynamic components. The weightless slider has a linear visco-elastic pair. As reported in [1, 2] the slider drifts in stick-slip phases where the relative oscillations between the mass and the slider are bounded and range from periodic to chaotic. The progressive motion of the mass occurs when the force acting on the slider exceeds the threshold of the dry friction force. $x,\ z,\ v$ represent the absolute displacements of the mass, slider top and slider bottom, respectively. It is assumed that the model operates horizontally, or the gravitational force is appropriately compensated. At the initial moment $\tau = 0$ there is a distance between the mass and the slider top called gap, e.

The considered system operates at the time in one of the following modes: *No contact*, *Contact without progression*, and *Contact with progression*. A detailed consideration of these modes and the dimensional form of equations of motion can be found in [1, 3]. As it was reported in [2] by introducing a new system of co-ordinates (p, q, v) instead of (x, z, v):

$$p = x - v, \qquad q = z - v, \tag{1}$$

it is possible to separate the oscillatory motion from the drift. In fact, in the new co-ordinates system p and q are displacements of the mass and the slider top relative to the current position of the slider bottom v. The phase space of the bounded system is shown in Figure 1(a). During *No contact* mode trajectory of the system lies in vicinity of the horizontal plane and during *Contact without progression* and *Contact with progression* modes it belongs to the inclined plane. The borders of the different modes are represented by lines $\Sigma_1 - \Sigma_4$. Equations of motion for this system are given below:

$$\begin{aligned}
p' &= y(1-\mathcal{H}_3) - \frac{1}{2\xi}(q-1)\mathcal{H}_1\mathcal{H}_3\mathcal{H}_4 \\
y' &= a\cos(\omega\tau+\varphi) + b - (2\xi y + q)\mathcal{H}_1\mathcal{H}_2(1-\mathcal{H}_3) - \mathcal{H}_1\mathcal{H}_3\mathcal{H}_4 \\
q' &= y\mathcal{H}_1\mathcal{H}_2(1-\mathcal{H}_3) - \frac{1}{2\xi}(q-1)\mathcal{H}_1\mathcal{H}_3\mathcal{H}_4 - \frac{1}{2\xi}q(1-\mathcal{H}_1)
\end{aligned} \tag{2}$$

where $\mathcal{H}_1 = H(p-q-e), \quad \mathcal{H}_2 = H(2\xi y + q), \quad \mathcal{H}_3 = H(2\xi + q - 1), \quad \mathcal{H}_4 = H(y).$

Here $H(\cdot)$ is Heaviside step function, y is the velocity of the mass, 2ξ is a damping ratio of the slider, $a\cos(\omega\tau+\varphi)$ is the dynamic and b is the static components of the external force. During *No contact* and *Contact without progression* phases the slider bottom remains stationary, $v' = 0$, and during *Contact with progression* phase it moves with velocity $v' = y + \frac{1}{2\xi}(q-1)$.

The equations of motion are linear for each phase, therefore the global solution can be constructed by joining the local solutions at the points of discontinuities. The set of initial values $(\tau_0; p_0, y_0, q_0)$ defines in which phase the system will operate. If $p_0 < q_0 + e$, it will be *No contact* phase. For $p_0 = q_0 + e$, it will be *Contact without progression* phase if $0 < 2\xi y_0 + q_0 < 1$ or *Contact with progression* phase if $2\xi y_0 + q_0 \geq 1$. When the conditions corresponding to the current phase fail, the next phase begins, and the final displacements and velocities for the preceding phase define the initial conditions for the next one. All details of the semi-analytical method allowing to calculate the responses of the system using this method are given in [2].

LOW DIMENSIONAL MAPS

The four dimensional flow $(\tau; p, y, q)$ can be locally three dimensional (during contact phases $p = q + e$). The cross-sections of the flow with the phase borders allows to formulate two dimensional discrete map. The borders of the different modes of the system shown in Figure 1(a) are given as:

$$\begin{aligned}
\Sigma_1 &= \{(\tau_i; p_i, y_i, q_i) |\; p_i = e,\; y_i > 0,\; q_i = 0\} \\
\Sigma_2 &= \{(\tau_i; p_i, y_i, q_i) |\; p_i = 1 + e - 2\xi y_i,\; y_i > 0,\; q_i = 1 - 2\xi y_i\} \\
\Sigma_3 &= \{(\tau_i; p_i, y_i, q_i) |\; p_i = 1 + e - 2\xi y_i,\; y_i < 0,\; q_i = 1 - 2\xi y_i\} \\
\Sigma_4 &= \{(\tau_i; p_i, y_i, q_i) |\; p_i = e - 2\xi y_i,\; y_i < 0,\; q_i = -2\xi y_i\}
\end{aligned} \qquad (3)$$

Based on the four subspaces (3), six local maps shown in Figure 1(a) can be defined as follows:

$$\mathbf{P}_1 : \Sigma_1 \to \Sigma_2, \quad \mathbf{P}_2 : \Sigma_2 \to \Sigma_3, \quad \mathbf{P}_3 : \Sigma_3 \to \Sigma_4, \quad \mathbf{P}_4 : \Sigma_4 \to \Sigma_1, \quad \mathbf{P}_5 : \Sigma_1 \to \Sigma_4, \quad \mathbf{P}_6 : \Sigma_3 \to \Sigma_2.$$

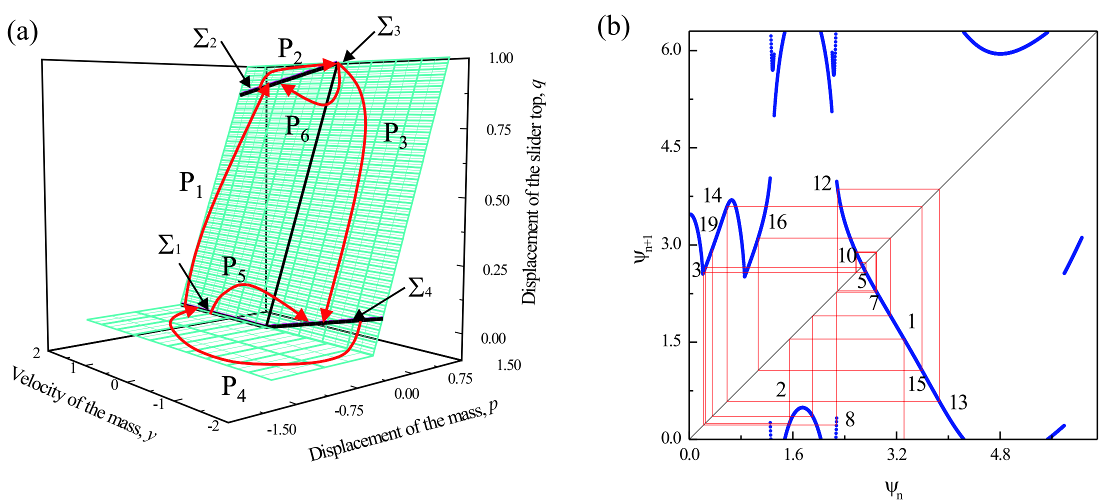

Figure 1. (a) Local mappings in three dimensional phase space; (b) iteration of the approximate one dimensional map experiencing chaos for $a = 0.3,\; b = 0.08,\; \xi = 0.01,\; \omega = 0.1,\; \varphi = 0,\; e = 0.02$.

At the beginning of the *Contact with progression* phase four dimensional flow crosses the line Σ_2 and system dynamics can be monitored using global two dimensional map $\mathbf{P} : \Sigma_2 \to \Sigma_2$ which maps velocity and angular displacement ($\psi = \omega\tau + \varphi$) at the beginning of the *Contact with progression* phase to themselves, $(y_{n+1}, \psi_{n+1}) = \mathbf{P}(y_n, \psi_n)$ [4]. Map $\mathbf{P}$ is unknown composition of the local maps, e.g it can be equal to $\mathbf{P} = \mathbf{P}_1 \circ \mathbf{P}_4 \circ \mathbf{P}_3 \circ \mathbf{P}_2$ or $\mathbf{P} = \mathbf{P}_6$ or $\mathbf{P} = \mathbf{P}_1 \circ \mathbf{P}_4 \circ \mathbf{P}_5 \circ \mathbf{P}_4 \circ \mathbf{P}_3 \circ \mathbf{P}_2$. For the periodic external force the introduced two dimensional map is defined in the bounded region $\psi_n \in (0, 2\pi),\; y_n \in (0, y^{max})$ and it can be calculated numerically [4].

The detailed analysis of the considered system reveals that a further reduction to 1D approximate map is possible. It has been found that the actual positions of the system at the end of the *Contact with progression* phase (points belonging to the subspace Σ_3) are very close to the points $\tilde{\Sigma}_3 = \{(\tau_i; p_i, y_i, q_i) |\; p_i = 1 + e,\; y_i = 0,\; q_i = 1\}$. In this case the approximate one dimensional map $\mathcal{F} : \tilde{\Sigma}_3 \to \tilde{\Sigma}_3$ can be introduced. The iteration of the proposed 1D approximate map for chaotic regime is shown in Figure 1(b).

References

[1] E. Pavlovskaia, M. Wiercigroch, and C. Grebogi: Modeling of an impact system with a drift, *Phys. Rev. E* **64**: 056224, 2001.

[2] E. Pavlovskaia and M. Wiercigroch: Analytical drift reconstruction for impact oscillator with drift, *Chaos, Solitons and Fractals* **19**: 151-161, 2004.

[3] E.E. Pavlovskaia and M. Wiercigroch, "Periodic solutions finder for vibro-impact oscillator with a drift", Journal of Sound and Vibration **267**, 893-911 (2003).

[4] E. Pavlovskaia and M. Wiercigroch: Two dimensional map for impact oscillator with drift, submitted in 2003.