TWO SCALE FINITE ELEMENT METHOD

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Summary A two scale computational method based on the finite element method is presented. A method is suited for problems with two length scales, macro and micro, where a direct numerical resolution of the micro scale by the standard finite element method is too expensive. At the micro scale a standard finite element method is used, while at the macro scale the Petrov-Galerkin method is used. Problems with several thousand cells and tens of the fibre orientations are readily computed.

INTRODUCTION

Modern materials have quite often two length scales, a macro scale which is proportional to the physical size of the object and a micro scale which is proportional to the fine structure of the material. Considering the material homogeneous at the macro scale its material properties are given by the effective material properties by the homogenization theory. In general, the homogenization theory gives only existence of the effective material properties and only in some particular cases, for example for laminated or periodic structure, gives explicit formula for the effective material properties in terms of the micromechanical material properties. Thus in general case to obtain effective material properties a numerical method must be used. To this end a new computational method is proposed.

DESCRIPTION OF THE METHOD

Partition of the variational problem

Let Ω be an open bounded subset of $\mathbb{R}^d$ with a piecewise regular boundary and let us consider the following macro scale boundary value problem in the variational form: find $u \in H_0^1(\Omega)$ such that $a(u,v) = (f,v)$ for all $v \in H_0^1(\Omega)$. Further, let $\{\Omega_k\}_{k=1,\dots,K}$ be a family of open pair wise disjoint bounded subsets of Ω with piecewise regular boundaries such that $\overline{\Omega} = \cup_{k=1}^{N} \overline{\Omega}_k$. Restriction of the form $a(\cdot,\cdot)$ to $H^1(\Omega_k) \times H^1(\Omega_k)$ is denoted by $a_k(\cdot,\cdot)$. To each set Ω_k we associate a reference domain $\mathcal{O}_{i(k)}$, $i(k) \in \{1,\dots,I\}$ and a diffeomorphisem $T_k : \mathbf{x}_{i(k)} \mapsto \mathbf{y}_k$ from $\mathcal{O}_{i(k)}$ onto Ω_k. It is expecting that $I << K$. For example, if Ω is made of identical cells, then $I = 1$ and each T_k is a rigid transformation. A mapping T_k pulls back form a_k to the form $a_k^*(\cdot,\cdot)$ defined on $H^1(\mathcal{O}_{i(k)}) \times H^1(\mathcal{O}_{i(k)})$. In particular, if $a(u,v) = \int_\Omega A\nabla u \cdot \nabla v \, d\mathbf{y}$, $A \in \mathbb{R}^{d\times d}$, then $a_k^*(u^*,v^*) = \int_{\mathcal{O}_{i(k)}} A_k^* \nabla u^* \cdot \nabla v^* \, d\mathbf{x}_{i(k)}$, where $A_k^* = \left|\frac{\partial \mathbf{y}_k}{\partial \mathbf{x}_{i(k)}}\right| \left(\frac{\partial \mathbf{x}_{i(k)}}{\partial \mathbf{y}_k} A \left(\frac{\partial \mathbf{x}_{i(k)}}{\partial \mathbf{y}_k}\right)^T\right)^*$. Next we define an equivalence relation $\mathcal{R}$ on the set $\mathcal{K} = \{1,\dots,K\}$ by $k_1 \,\mathcal{R}\, k_2 \iff (A_{k_1}^* = A_{k_2}^* \quad \& \quad i(k_1) = i(k_2))$ and a factor set $\mathcal{K}_0 = \mathcal{K}/\mathcal{R}$. It is expecting that $K_0 = |\mathcal{K}_0|$ is a small number comparing to K. For example, if Ω is a square domain of the unit size and Ω_k are the squares with the sides of length $\frac{1}{n}$, $n \in \mathbb{N}$, aligned with the sides of Ω then $K = n^2$, $I = 1$ and each T_k is a translation from $\mathcal{O}$ to Ω_k. Further, if restriction of A to Ω_k represents material properties of Ω_k and each Ω_k is a composite made of the matrix Ω_k^{m} and the inclusion Ω_k^{i} with the material coefficients $C_{j(k)}^{\mathrm{m}}$ and $C_{j(k)}^{\mathrm{i}}$, $j \in \{1,\dots,J\}$, respectively, then $K_0 = \sum_{j=1}^{J} m(j)$, where $m(j)$ is a number of different geometrical arrangements of the inclusions Ω_k^{i} with the same material coefficients $C_{j(k)}^{\mathrm{i}}$. Typically, K_0 is of order 10 and K could be of order 10^4 or even of greater order.

Macro and micro finite element discretization

Let $\mathcal{T}_H$ be a finite element triangulation of Ω subordinate to the partition $\overline{\Omega} = \cup_{k=1}^{N} \overline{\Omega}_k$. In another words, each $\overline{\Omega}_k$ is a finite union of the sets from $\mathcal{T}_H$. The corresponding finite element space with the zero boundary data is denoted by V_{0H} and the basis functions by Ψ. The space V_{0H} is called the macro finite element space. Further, we denote by $\mathcal{T}_{h_k}$ a finite element triangulation of Ω_k and by V_{h_k} the corresponding finite element space. The spaces V_{h_k} are called the micro finite element spaces. It is required that the restrictions of the macro finite element space V_{0H} to Ω_k are contained in the micro finite element spaces V_{h_k}. In another words, micro finite element discretization is finer than the macro element discretization. On each Ω_k we associate to every macro basis function Ψ_p a discrete microscale variational problem : find $M_{p,k}^1 \in V_{h_k}$ such that $M_{p,k}^1 = \Psi_p$ on $\partial\Omega_k$ and $a_k(M_{p,k}^1, \phi) = (f_k, \phi)$ for all $\phi \in V_{h_k}$ and a variational problem independent of Ψ_p: find $M_k^0 \in V_{0h_k}$ such that $a_k(M_k^0, \phi) = (f_k, \phi)$ for all $\phi \in V_{0h_k}$. Here V_{0h_k} is a subset of V_{h_k} with the zero boundary data along $\partial\Omega_k$. For given $M_{p,k}^1$ and M_k^0 we define microscale basis function $M_{p,k} = M_{p,k}^1 + (\alpha_k - 1)M_k^0$ where α_k is yet unspecified parameter. Global microscale basis functions M_p are defined over Ω by combining definitions of $M_{p,k}$ over the supports of Ψ_p and extending by zero outside of them. It should be noted that it suffices to actually compute microscale basis functions only for the indices $k_0 \in \mathcal{K}_0$ since for $k \notin \mathcal{K}_0$ microscale basis functions are obtained by the mapping $T_k \circ T_{k_0}^{-1}$.

Having found microscale basis function we approximate solution of the macro scale variational problem by the linear combination $u_{H,h} = \sum_{p=1}^{P} u_p M_p$ such that 1) $u_{H,h}$ is a Petrov-Galerkin solution for the finite element triangulation $\mathcal{T}_H$ with the coordinate solutions M_p and the weighting functions Ψ_p and 2) restriction of $u_{H,h}$ to Ω_k is a solution of the microscale variational problem. It follows from the second condition that $\sum_{p\in P(k)} u_p \,\alpha_k = 1$, where $P(k) = \{p :$

supp $\Psi_p \cap \Omega_k \neq \emptyset\}$. Using this equation in the Petrov-Galerkin statement $a(u_{H,h}, \Psi) = (f, \Psi)$ for all $\Psi \in V_{0H}$ we obtain a system of linear equations

$$\sum_{p \in P(q)} \sum_{k \in K(p)} u_p\, a_k(M^1_{p,k} - M^0_k, \Psi_q) = (f, \Psi_q) + \sum_{k \in K(p)} u_p\, a_k(M^0_k, \Psi_q), \qquad (1)$$

$q = \{1, \ldots, P\}$, where $P(q) = \{p : \text{supp}\, M_p \cap \text{supp}\, \Psi_k \neq \emptyset\}$ and $K(p) = \{k : \text{supp}\, M_p \cap \Omega_k \neq \emptyset\}$. Rewriting (1) in the compact notation we have $\mathbf{Bu} = \mathbf{b}$. Note that the components B_{qp} of $\mathbf{B}$ and b_q of $\mathbf{b}$ are expressible as the linear combinations of $a_k(\phi_{kr}, \phi_{ks})$ and (f, ϕ_{kr}), where $\phi_{kr}, \phi_{ks} \in V_{h_k}$ have nonzero boundary values along $\partial\Omega_k$. The matrix $\mathbf{B}$ is sparse but in general unsymmetric. If the bilinear form $a(\cdot, \cdot)$ is coercive, the system $\mathbf{Bu} = \mathbf{b}$ has a unique solution. Due to the construction convergence of the macro finite element is $O(H^\beta)$, where the order of convergence β is given by the global regularity of the solution and the interpolation properties of V_{0H}. It is expected but not yet proved that the solution has certain superconvergence properties of order $O(h^\gamma)$.

Validation of the method

A simple test to validate the two scale finite element computation is to apply the method to the problem with the homogeneous material. To this end, let for a given macro finite element discretization a partition of Ω be coincident with the triangulation $\mathcal{T}_H$. Then the maro/micro finite element solution $u_{H,h}$ has two scale convergence properties, namely, for a fixed h we have $\lim_{H \to 0} u_{H,h} = u$ and for a fixed H we have $\lim_{h \to 0} u_{H,h} = u$. Choosing appropriate mesh parameters H and h a significant reduction of the computational time and space, comparing to the global mesh discretization of the size h can be achieved. Another test is provided by the homogenization theory by comparing solution of the problem with the effective material properties and the solution of the problem with highly oscillating periodic material properties. The proposed two scale finite element method passed both two tests by the predicted accuracy.

NUMERICAL EXAMPLES

Due to the restricted space only a very simple numerical example is given. For Ω is chosen a unit square which is partitioned into N^2 squares Ω_k with the area $\frac{1}{N^2}$. Each Ω_k is a bimaterial composite made of the isotropic matrix and the isotropic elliptical inclusion with a random orientation α_k where α_k is the angle between the major axis of the ellipse and the direction of the chosen side of the square Ω. We assume that α_k take values from the set $\{\alpha_1, \ldots, \alpha_{K_0}\}$ with the same probability. The semi axis of the ellipse are $\frac{a}{N}$ and $\frac{b}{N}$ with the nondimensional numerical values $a = 0.4$ and $b = 0.2$. The nondimensional material matrix of the matrix Ω_k^{m} is the identity matrix $\mathbf{I}$ and the nondimensional material matrix of the inclusion is $\beta\mathbf{I}$. As found by the numerical computation, weak inclusion gives more pronounced dependance upon the orientation and thus results for $\beta = 10^{-3}$ are presented. The nondimensional load vector is $f = 2$. Summarizing, the problem is to find a solution $u \in H^1_0(\Omega)$ such that $\int_\Omega \hat{\beta}, \nabla u \cdot \nabla v \, d\Omega = 2 \int_\Omega v \, d\Omega$ for all $v \in H^1_0(\Omega)$. Here $\hat{\beta} = 1$ on Ω_k^{m} and $\hat{\beta} = \beta$ on Ω_k^{i}. In terms of mechanics, u is either a Prandtl stress function of a torsion problem for a composite cylindrical bar or either a solution of the steady state heat equation with the internal heat source f, see Fig. 1. In the first case nondimensional torsional rigidity $D = 2 \int_\Omega u \, d\Omega$ is of interest, see Table 1.

CONCLUSIONS

The proposed new two scale finite element computation is highly effective. It is capable to accurately resolve the micro scale at the cost of the macro element discretization. Application of the method to the problems with the homogeneous material is considered as an important improvement of the finite element method.

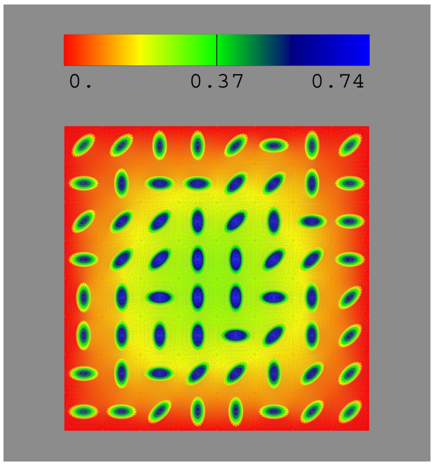

Table 1. Dependance of the torsional rigidity upon the number of cells $K = n^2$; D_{p} torsional rigidity of the periodic structure with two orientations $\alpha_1 = 0$ and $\alpha_1 = 90°$, D_{u} torsional rigidity of the unidirectional periodic structure with $\alpha_1 = 0$. Element type : quadratic micro element.

n	h	H	D_{p}	D_{u}
2	$1/16n$	h	2.21001	2.21200
4	$1/16n$	h	0.73591	0.73895
8	$1/16n$	h	0.37091	0.37432
16	$1/16n$	$2h$	0.28006	0.28343
32	$1/16n$	$2h$	0.25747	0.26076
64	$1/16n$	$4h$	0.25147	0.25443
128	$1/16n$	$16h$	0.24872	0.24921

Fig. 1. Temperature distribution within the media with the random oriented inclusions.