STABILITY OF A ROTOR WITH PERIODICALLY VARYING ANGULAR VELOCITY

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Summary The stability of a rotor with periodically varying angular velocity is investigated. The stability region in the parameter space is determined, which appears to be remarkably large. Parametric resonances are not found in this case.

INTRODUCTION

In the past, the dynamics of rotors that spin at a nominally constant rate have been studied and their stability has been investigated. An overview has been given by Crandall [1]. The influence of want of balance, asymmetry in stiffness and mass distribution, and viscoelastic material behaviour has been studied. Furthermore, the processes of starting and stopping of a rotor and the transition through critical speeds have been included in the analysis [2]. Less attention has been paid to the case in which the angular velocity is varied periodically, the case of parametric excitation. In the present study, in particular, the case in which the variations of angular velocities are large and the frequency of the excitation is low in comparison with the frequency of rotation will be considered.

As the simplest system, a massless simply supported shaft with equal bending stiffness in all directions and with a disk in its centre is studied; this system is usually called a Jeffcott rotor [3]. The shaft is rotated at a rate Ω that is a prescribed function of time. The dimensionless equations of motion, written in complex variables, with $z = x + iy$, where x and y are the lateral displacements of the centre of the disk, are

$$\ddot{z} + 2(c_\mathrm{e} + c_\mathrm{i})\dot{z} + (1 - 2i\Omega c_\mathrm{i})z = z_0(\Omega^2 - i\dot{\Omega})e^{i\theta}. \tag{1}$$

Here, c_e and c_i are the external and internal relative damping respectively, $z_0 = x_0 + iy_0$ is the mass eccentricity, and θ is the rotation angle. Extended models can include more general kinds of viscoelastic material behaviour for the shaft and different bending stiffnesses in two directions. For a stability analysis, only the homogeneous equations need be studied.

STABILITY ANALYSIS OF THE ROTOR

The stability analysis leads to the transformed equation

$$\ddot{z} + [1 - (c_\mathrm{e} + c_\mathrm{i})^2 - 2i\Omega c_\mathrm{i}]z = 0. \tag{2}$$

The stability condition is now that the real parts of the characteristic exponents must be smaller than $c_\mathrm{e} + c_\mathrm{i}$.

First the results for a rotor that spins at a constant angular velocity are recalled to mind. As has been found experimentally and theoretically by Newkirk and Kimball [4, 5], the internal damping makes the rotor unstable above a limit speed. For the system in Eq. (1), this speed is $\Omega_\mathrm{l} = 1 + c_\mathrm{e}/c_\mathrm{i}$.

For piecewise constant Ω, we are led to an equation studied by Meissner [6], but in the complex domain with complex parameters. The equations for the stability boundaries based on Floquet's theory [7] can be explicitly calculated. In the case that $\Omega = \Omega_0$ during half of the period and $\Omega = -\Omega_0$ during the other half, the expressions remain manageable. For small damping and large rotation rates, these yield equations for the stability boundaries of the form

$$T_\mathrm{p}\sqrt{\Omega_0 c_\mathrm{i}} = O(1). \tag{3}$$

where T_p is the period of the parametric excitation.

It is seen that if the internal damping is small, which is usually the case, the rotor remains stable for quite large values of the angular velocity or large periods of the parametric excitation. In particular, the equations with frozen coefficients can be unstable for all times, while the frequency of the variation of the angular velocity is relatively low.

For the more important case of a harmonically varying angular velocity, $\Omega = \Omega_0 \cos(2\pi t/T_\mathrm{p})$, Mathieu's equation [8] with complex coefficients is obtained. Although the stability conditions cannot be found in closed form, the stability boundaries in the parameter space can be determined either from tabulated values of the characteristic exponents or from numerical calculations. Qualitatively, the behaviour is the same as for the case with piecewise constant angular velocity.

INFLUENCE OF VISCOELASTICITY AND UNEQUAL BENDING STIFFNESSES

If the viscous internal damping is replaced by viscoelastic damping of the Maxwell type with a small relaxation time, which gives the so-called standard linear solid for the shaft [9], the limit speed increases a little and, more importantly, stability is regained at high constant angular velocities. This has as a consequence that for a fixed period of the parametric excitation, the rotor may be stable for any amplitude. In general, the region of stability is increased.

Small differences in the bending stiffness in two perpendicular directions have little influence on the results, although large differences may lead to instability.

CONCLUSIONS

It has been found that a rotor that undergoes a periodic variation of its velocity of rotation shows a remarkably large region of stability, even if the system with a corresponding fixed velocity of rotation is unstable for most of the time. Regions of parametric resonance are not found, which contrasts to most other examples of parametric excitation. For a viscoelastic material behaviour, the region of stability is even further enlarged. These results might be of some importance for rotors for which a constant angular velocity is not needed.

References

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