NONLINEAR THREE-DIMENSIONAL FREE SURFACE FLOWS IN FINITE AND INFINITE DEPTH

Emilian Părău*, Jean-Marc Vanden-Broeck*, Mark J. Cooker*
*School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, UK

Summary Boundary integral equation methods are used to calculate steady three-dimensional potential free surface flows. The formulation uses Green's functions and generalises the work of Forbes [1] to finite depth. Accurate numerical results are presented for free surface flows generated by moving disturbances such as distributions of pressure, ships and submerged objects.

INTRODUCTION

In the last hundred years, important progress has been achieved in the calculation of two dimensional free surface flows. Numerical solutions to nonlinear three dimensional potential free surface flows have also been calculated, but most of them solve model equations or use algorithms which solve the time dependent formulation of the problem. An extensive review of the computations of nonlinear free-surface flows is given by Tsai & Yue [2]. Here iterative methods are presented to solve directly the steady fully nonlinear problem.

FORMULATION

We present the method in the case of a three-dimensional distribution of pressure moving at a constant velocity U at the surface of a fluid of finite depth h. We choose a frame of reference moving with the pressure distribution and assume that the flow is steady. We introduce cartesian coordinates x, y, z with the z-axis directed vertically upwards and the x-axis in the opposite direction of the velocity U. We denote by $z = \zeta(x, y)$ the equation of the free surface. The potential function $\Phi(x, y, z)$ satisfies Laplace equation

$$\nabla^2 \Phi = 0, \qquad x, y \in \mathbf{R}, z < \zeta(x, y), \tag{1}$$

in the flow domain.

The kinematic boundary condition, the dynamic boundary condition, the bottom boundary condition and the radiation condition can be written as

$$\Phi_x \zeta_x + \Phi_y \zeta_y = \Phi_z, \quad z = \zeta(x, y), \tag{2}$$

$$\frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + g\zeta + \frac{p}{\rho} = \frac{U^2}{2}, \quad z = \zeta(x, y), \tag{3}$$

$$\Phi_z = 0 \qquad \text{on } z = -h. \tag{4}$$

$$\text{no waves} \quad \text{as} \quad x \to -\infty. \tag{5}$$

Green's second identity holds in three dimensions, where V represents a volume bounded by the surface δV

$$\int_V (\alpha \Delta \beta - \beta \Delta \alpha) dV = \int_{\delta V} (\alpha \frac{\partial \beta}{\partial n} - \beta \frac{\partial \alpha}{\partial n}) ds. \tag{6}$$

By using the three-dimensional Green function which satisfies (4)

$$G(P, Q) = \frac{1}{4\pi} \left(\frac{1}{((x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2)^{1/2}} + \frac{1}{((x - x^*)^2 + (y - y^*)^2 + (z + z^* + 2h)^2)^{1/2}} \right) \tag{7}$$

where $P = (x, y, z), Q = (x^*, y^*, z^*)$, we obtain

$$2\pi(\Phi(Q) - x^*) = \int\int_{S_F} ((\Phi(P) - x) \frac{\partial G(P, Q)}{\partial n_P} - G(P, Q) \frac{\partial}{\partial n_P} (\Phi(P) - x) \, ds_P \tag{8}$$

where S_F is the free surface. This equation is projected in the $x - y$ plane.

We present results by choosing the pressure as $p(x, y) = \begin{cases} P_0 e^{\frac{L^2}{(x^2 - L^2)} + \frac{L^2}{(y^2 - L^2)}}, & |x| < L \text{ and } |y| < L \\ 0, & \text{otherwise} \end{cases}$.

We introduce dimensionless variables by using U as the unit velocity and L as the unit length. Combining equations (2) and (3) and using the chain rule of calculus we obtain

$$\frac{1}{2} \frac{(1 + \zeta_x^2)\phi_y^2 + (1 + \zeta_y^2)\phi_x^2 - 2\zeta_x \zeta_y \phi_x \phi_y}{1 + \zeta_x^2 + \zeta_y^2} + \frac{\zeta}{F_L^2} + \varepsilon P = \frac{1}{2} \tag{9}$$

where $F_L = U/(gL)^{1/2}$ and $\varepsilon = \frac{P_0}{\rho U^2}$. When the depth is finite we use the Froude number based on depth $F = U/(gh)^{1/2} = F_L/\sqrt{H}$ where $H = h/L$.

The singularities in the equation (8) can be isolated and evaluated in closed form (see Forbes [1]). The numerical sheme to solve the equations (9) and (8) is similar to the one used in Părău and Vanden-Broeck [3], where the problem is discretized by using finite differences and the resulting algebraic equations are solved by Newton's method. Extensions of the numerical procedure to other disturbances (ships and submerged objects) will be discussed in the talk.

RESULTS

Various numerical results will be presented. We include here examples for the free surface flow generated by a moving distribution of pressure. In infinite depth the wake and the two different families of waves (transverse waves and short-length divergent waves) can be easily observed. The transverse waves are perpendicular to the direction of the velocity (i.e. the x-axis). The divergent waves have crests roughly parallel to the direction of velocity, moving outward. When F_L increases the amplitude of the divergent waves becomes more important than that of the transverse waves. The wavelength of the transverse waves increases with the Froude number. Nonlinear solutions can be calculated close to the maximum height of waves allowed by Bernoulli's equation.

In finite depth, when $F < 1$, the wave–pattern is similar to the case of infinite depth, but when $F > 1$ the transverse waves disappear and the angle of the outer waves decreases with F (see Fig. 1).

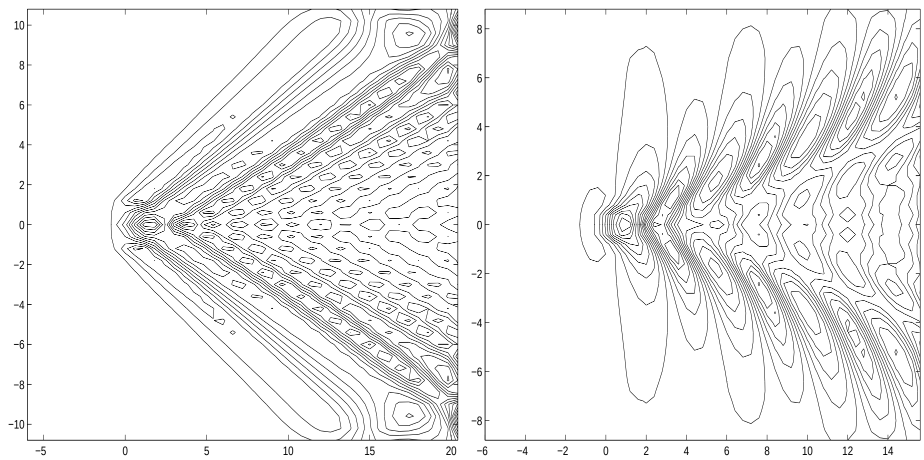

Figure 1: The wake in the cases $F > 1$ (left) and $F < 1$(right). In both cases $\varepsilon = 1$.

The algorithm can be easily modified to include two or more pressure distributions and to study the interaction of the wakes produced by each of them. The V-shape of the waves downstream becomes in that case a W-shape. This can be viewed as the wave interactions between ships moving in parallel in deep water.

CONCLUSION

We have calculated three-dimensional free surface flows generated by moving disturbances in a fluid of finite or infinite depth. Accurate fully nonlinear solutions have been obtained.

References

[1] L. K. Forbes, An Algorithm for 3-Dimensional Free-Surface Problems in Hydrodynamics, *Journal of Computational Physics* 82 (1989) 330-347.

[2] W. T. Tsai, D. K. Yue, Computation of nonlinear free-surface flows, *Ann. Rev. Fluid Mech.* 28 (1996) 249-278.

[3] E. Părău, J.-M. Vanden-Broeck – Nonlinear two- and three- dimensional free surface flows due to moving disturbances, *Eur. J. Mechanics B/Fluids*, **21**, 6 (2002), 643-656.