XXI ICTAM, 15–21 August 2004, Warsaw, Poland

DAM-BREAK FLOW FOR ARBITRARY SLOPE OF THE BOTTOM

R. Fernandez-Feria
E.T.S. Ingenieros Industriales, University of Málaga, Spain

Summary The dam-break flow problem in the shallow-water approximation on an inclined bed for arbitrary slopes of the bottom is considered. An analytical solution for the spreading of the water fronts at the initial stages is given. A self-similar solution asymptotically valid for large times is also found. For intermediate times the problem is solved numerically by the method of characteristics.

FORMULATION OF THE PROBLEM

Consider the one-dimensional shallow-water equations for the flow over a constant slope bed,

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta U}{\partial X} = 0\,, \tag{1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + \cos\theta\frac{\partial \eta}{\partial X} = \sin\theta\,, \tag{2}$$

where θ is the angle between the bed and the horizontal, t is the time, X is the coordinate along the bed, η is the depth of the water measured along the coordinate Y perpendicular to the bed, and U is the depth-averaged velocity component along X [see Fig. 1]. All the magnitudes in the above equations are dimensionless, and have been non-dimensionalized with respect to a length scale η_0, corresponding to some initial depth, and a velocity scale $U_0 = \sqrt{g\eta_0}$, where g is the acceleration due to gravity. These equations can be derived from the Euler equations for an incompressible fluid under the assumption that the characteristic length scale of the flow in the direction of the coordinate X is much greater than the characteristic length scale in the Y direction. According to [1], equations (1)-(2) are valid for any slope $\tan\theta$ of the constant-slope bed, provided that the above assumption is satisfied.
We are interested here in solving these equations for the dam-break problem, i.e., for the flow whose initial condition ($t = 0$) is given by (see Fig. 1)

$$U(0, X) = 0\,, \tag{3}$$

$$\eta(0, X) = \begin{cases} 0 & \text{for } X < -1/e \\ eX + 1 & \text{for } -1/e \le X \le 0 \\ -X/e + 1 & \text{for } 0 < X \le e \\ 0 & \text{for } X > e \end{cases}\,, \tag{4}$$

where $e \equiv \tan\theta$ is the slope of the bed. At $t = 0$, the vertical wall that intersects the bed at $X = e$ is removed instantaneously, causing the fluid to move over the sloping bed under the action of gravity. Note that characteristic length η_0 is the dimensional depth at $X = 0$, $t = 0$.

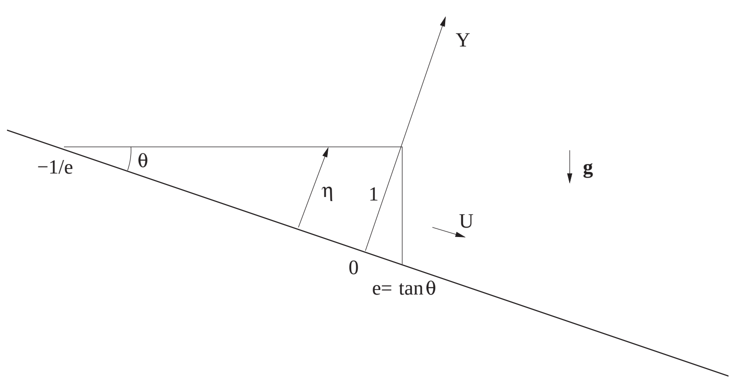

Figure 1. Coordinates and sketch of the initial conditions.

RESULTS

Analytic solution for the advance of the water fronts at the initial stages

Analytic solutions for the advance of the left and right water fronts at the initial stages are found. In fact, it is found that the initial condition near the left front, i.e. $U = 0$, and $\eta = 0$ for $X < -1/e$, $\eta = 1 + eX$ for $X > -1/e$, remains valid in the region of the (t, X) plane $-1/e \leq X \leq -\sqrt{\cos\theta}\, t$, which means that the left water front remains at rest for $t \leq t_2 \equiv 2\sqrt{\cos\theta}/\sin\theta$. On the other hand, the solution for the advance of the right water front at the initial stages is given by

$$U(t, X) = \frac{t}{\sin\theta}\,, \tag{5}$$

$$\eta(t, X) = \begin{cases} 0 & \text{for } X > e + \frac{1}{\sin\theta}\frac{t^2}{2} \\ -\frac{X}{e} + 1 + \frac{1}{e\sin\theta}\frac{t^2}{2} & \text{for } X < e + \frac{1}{\sin\theta}\frac{t^2}{2} \end{cases}\,, \tag{6}$$

valid in the region of the (t, X) plane $\sqrt{\cos\theta}\, t + [\cos\theta/(2e) + \sin\theta]t^2/2 \leq X \leq e + t^2/(2\sin\theta)$, $t \leq t_1 \equiv 2e/\sqrt{\cos\theta}$. In addition to provide an useful information about the initial water spreading after the breaking of the dam, the above solutions allow the starting of the numerical integration of the equations by the method of characteristics near the singular points $(t = 0, X = -1/e)$ and $(t = 0, X = e)$. Figure 2 shows several profiles of the water depth at different times obtained from the above solutions and the method of characteristics for $e = 0.2$ ($\theta \simeq 11.31^o$; note that for this slope, $t = 1.5$ is between $t_1 \simeq 0.4039$ and $t_2 \simeq 10.0985$).

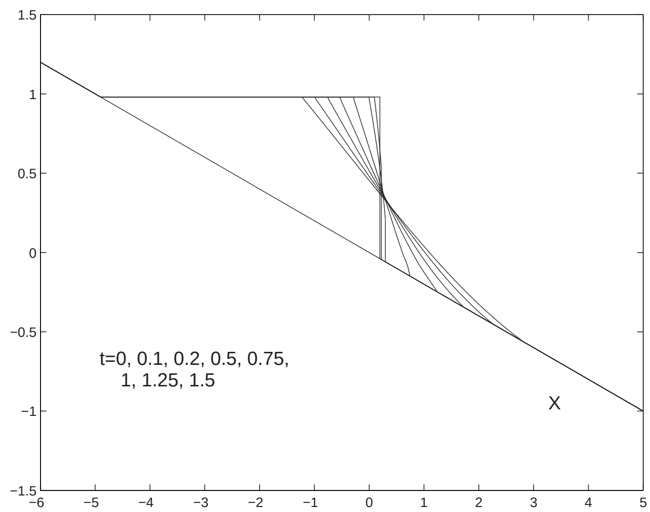

Figure 2. Profiles of the water depth along the sloping bed for $e = 0.2$, and for several instants of time, as indicated

Self-similar solution for large times

For $t \to \infty$, a self-similar solution of the second kind (see [2] for the terminology) is found:

$$\eta(t, X) = \frac{3(\xi - \xi^2)}{\sin\theta\cos\theta\,(u_m t + l_0)}\,, \tag{7}$$

$$U(t, X) = \sin\theta\, t + u_m\,[1 + \phi(t)]\left(\xi - \frac{1}{2}\right)\,, \quad \phi(t) = \frac{6}{u_m \sin\theta\,(u_m t + l_0)}\,, \tag{8}$$

where the self-similar variable is

$$\xi = \frac{X + (u_m t - \sin\theta\, t^2)/2 - l_{02}}{u_m t + l_0}\,, \qquad 0 \leq \xi \leq 1\,, \tag{9}$$

and the constants u_m, l_0, and l_{02} have to be obtained numerically for each initial condition (they depend only on θ; for instance, for $e = \tan\theta = 0.2$, it is found that $u_m \simeq 3.858$, $l_0 \simeq -3.381$, and $l_{02} \simeq 5.191$). At each instant t, the total water extent along X is given by $0 \leq \xi \leq 1$.

References

[1] Keller, J.B.: Shallow-water theory for arbitrary slopes of the bottom. *J. Fluid Mech* **489**:345–346, 2003.
[2] Barenblatt, G.I.: Scaling, self-similarity, and intermediate asymptotics. Cambridge U.P., Cambridge 1996.