# GENERALIZED CONTINUUM MECHANICS : THREE PATHS

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*Summary* The material framework is considered to place in evidence three essential ways of generalizing standard continuum mechanics, the latter being the theory of one-component simple materials (after W.Noll) with symmetric Cauchy stress. The three possible paths to generalization are (i) the loss of the Euclidean nature of the background material manifold, (ii) the loss of validity of Cauchy's construct of the notion of stress and (iii) the loss of symmetry of the latter. Special attention is paid to the consequences of these different losses on the canonical balance of momentum and its moment, whose ontological status is the same as the balance of energy, i.e., they concern the whole physical system under consideration (in particular, all degrees of freedom simultaneoulsy). Because of (i), these considerations of necessity are in that material framework which is the realm of configurational forces and forces driving structural defects.

## BACKGROUND

With the advent of homogenization techniques, the inclusion of metallurgical -and other - microstructures, the production of artificial materials with a controlled microstucture, and a growing interest in continuum mechanics at a smaller scale, the question of what is a generalized continuum comes up naturally. Here we would like to ponder the general notion of " generalized continuum " in the light of our thirty five years involvment in the field and the more recent developments in the so-called " material " mechanics of materials where the Eshelby stress is the main force-like ingredient (a longer paper in French is Ref.[1]).

What is to be understood by " generalized continuum mechanics " ? It obviously is dangerous, and somewhat preposterous to define the degree of generalization of a theory. For instance, many mathematicians and engineers called " generalized Hooke law " the constitutive equation for a linear , homogeneous but *anisotropic* elastic material. we shall all admit that this is a very weak generalization. The same holds true of the Hooke-Duhamel stress constitutive equation in *linear* thermoelasticity. The situation still is of the same type is the now again fashionable (because of the consideration of so-called smart materials) linear theory of *piezoelectricity* since not only the required anistropy can be dealt with easily but many problems can be reformulated using a four-dimensional " displacement " (being the classical one to which is adjoined the electrostatic potential). This is true because of the smallness of the electric fields involved. The situation is totally different when one deals with heterogeneous elasticity where a characteristic length introduces itself in the formulation, rendering dynamical wave-like problems *dispersive*. In all these cases, though, the Cauchy stress tensor remains *symmetric*.

## THE THREE PATHS

The first temptation to call a continuum theory really *generalized* comes with the consideration of a *non-symmetric* Cauchy stress tensor , e.g., when the ponderomotive couple due to electromagnetic fields in finitely electrically polarizable or magnetizable bodies is no longer neglegible. This is documented in detail in Ref.[2] as well as the much more complicated case when electric and magnetic degrees of freedom are associated with a corresponding microstructure (due to electric or magnetic dipoles, such as in ferroelectrics and ferromagnets of different types, etc). Note only is the Cauchy stress tensor no longer symmetric but then the basic Euler-Cauchy equation of motion (balance of physical linear momentum) is coupled to additional local balance equations related to these microstructures. This introduces the notion of additional degrees of freedom, albeit of a nonmechanical nature. Clearly, the introduction of additional degrees of freedom of a mechanical nature a each material point will produce the same type of generalization. This occurs when one recognizes that the kinematic-deformation modelling of a material point $\mathbf{X}$ by the usual degree of translation (the " displacement " giving rise by spatial differentiation to the notions of deformation and classical rotation) is not sufficient to describe the clearly present microstructure of many materials. The relevant concepts go back to Duhem and the Cosserat brothers, but they were revisited in the 1960s by many authors (Aero and Kuvshinskii, Palmov, Grioli, Eringen and Suhubi, Toupin, Green and Rivlin, Mindlin and Tiersten, W.Nowacki, etc). Two types of approaches are to be put forward, one attributing a set of *directors* , rigidly or nonrigidly attached to one another, to each material point (the Duhem-Toupin-Ericksen way) and another one introducing at each material point $\mathbf{X}$ a micro-deformation of a purely rotating type (the Cosserat or micropolar model) or of a general deformation type (so-called micromorphic model of Eringen *et al* [3]). All these models are conveniently introduced without ambiguity by using the principle of virtual power as the essential tool of formulation [4]-[5].

Two other steps of going further toward generalization consist in considering first a *weak*, and then a *strong* nonlocality in the *internal force* effects of which stress is an example. The first of these consists in introducing successive gradients of a field as a priori independent field variables at the same point **X** (the idea of a Cauchy expansion of a field). This in principle is applicable to all fields but the displacement is that field considered in the purely mechanical case. Noll's notion of a *simple material* is lost as well as the traditional Cauchy construct for the introduction of the stress notion [6]-[7]. The situation is worst for a strongly nonlocal theory in which the mechanical response at a material point **X** theoretically depends on what occurs kinematically at all points in the body [8]. The argument is strictly valid only for an infinite body, so that in truth the notion of Cauchy stress itself is lost. Finally, in all previous examples we assumed that the material manifold is a classical Euclidean one. This hypothesis may be given up by choice or necessity in the presence of continuous distributions of defects which hinder the continuity of the displacement field. This is the Bilby-Kroener-Noll-Wang-Epstein-Maugin-Rakotomanana (and others) way.

From the above, putting now the arguments in the logical order, we can propose a definition of " generalized continuum mechanics " : We say that we have such a mechanics whenever one of the following three basic tenets of classical continuum mechanics is no longer true : (i) the background material manifold is Euclidean, (ii) Cauchy's construct of the notion of stress holds true, (ii) Cauchy's stress is symmetric. Because of point (i) above the present contribution examines consequences of the loss of anyone of the hypotheses directly on the *material manifold*, where the appropriate stress measure is Eshelby's material stress tensor and this is the realm of all kinds of true or pseudo material inhomogeneities, as emphasized in many recent works [9]-[11]. Special attention is paid to the local *canonical* balance of momentum and its moment that accompany the balance of energy. All considerations are developed in comparison with the somewhat standard purely elastic, although nonlinear anisotropic, inhomogeneous case. Not only does this capture all the fields introduced but, as is known, it also permits constructive considerations in the theory of field singularities.

**References**


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