NUMERICAL DETECTION AND CONTINUATION OF SLIDING BIFURCATIONS IN A DRY-FRICTION OSCILLATOR

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Summary We will discuss a novel numerical method for the detection and continuation of codimension-1 sliding bifurcations of limit cycles in piecewise-smooth dynamical systems. A dry-friction oscillator is used as a representative example of relevance in applications.

INTRODUCTION

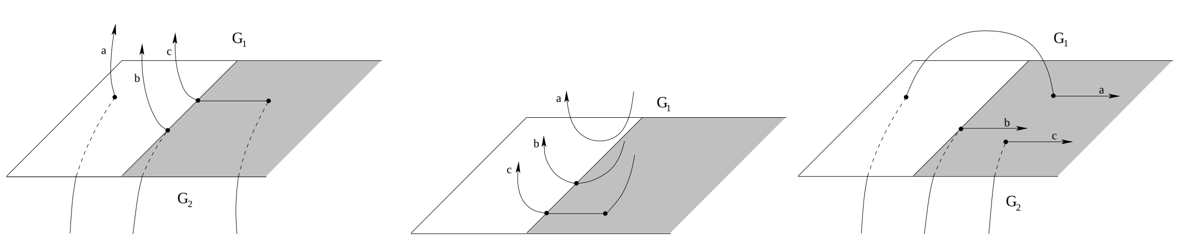
In this paper we focus our attention on numerical investigations of *sliding bifurcations* which are non-standard bifurcations intrinsic to systems modelled by a set of ordinary differential equations with discontinuous right hand side, which we can write as $\dot{x} = F_1(x, \mu)$ for $H(x) > 0$ and $\dot{x} = F_2(x, \mu)$ for $H(x) < 0$, where F_1, F_2 are sufficiently smooth vector functions and $H(x)$ is some scalar function depending on the system states. Such systems are known as *Filippov systems* [4, 7] and often model dry-friction oscillators. Discontinuous friction characteristics gives rise to a vector field which is discontinuous across some region in the phase space.

DRY-FRICTION OSCILLATOR

We focus our attention on a representative model of dry friction oscillator, which in the non-dimensionalised form can be written as

$$\ddot{x} + x = \sin(\omega t) - F\mathrm{sgn}(\dot{x}), \tag{1}$$

where x is the position, $\dot{x}$ the velocity, and t the time while ω and F respectively represent the frequency and the amplitude of the forcing term. The discontinuity set where the vector field switches between F_1 and F_2 in (1) is defined as $\Sigma := \{(x, \dot{x}, t) \in \mathbb{R}^3 : H(x, \dot{x}, t) = \dot{x} = 0\}$. An intriguing feature of Filippov systems is the possibility of exhibiting solutions evolving within Σ, termed as *sliding motion.* Sliding corresponds to a stick phase in the dry-friction oscillator dynamics of interest [6]. We can define a vector field, say F_s governing the stick phase of the system dynamics (sliding motion) using Utkin's equivalent control method [7]. This method also allows to determine a sliding subset, say $\hat{\Sigma}$ where the system dynamics is governed by the vector field F_s. Existence of the sliding subset $\hat{\Sigma}$ may qualitatively influence the system dynamics through the occurrence of sliding bifurcations. Sliding bifurcations are events due to the interaction of a system trajectory with the boundary of the sliding region. As shown in [3], there are four distinct codimension-1 sliding bifurcation scenarios of limit cycles namely: *crossing sliding* (Fig. 1(a)), *grazing sliding* (Fig. 1(b)), *switching sliding* (Fig. 1(c)) and *adding sliding* (not depicted). Sliding bifurcations can organise complex dynamics. For example in [1] it was shown that the grazing-sliding bifurcations scenario was responsible for the onset of chaotic stick-slip mode of motion in the dry-friction oscillator. Therefore, it is important to be able to detect and continue branches of sliding bifurcations. The theory allowing analysis of codimension-1 sliding bifurcations of limit cycles as well as of the so-called degenerate codimension-2 sliding bifurcation which are organising centres for branches of codimension-1 sliding bifurcations was developed recently in [2, 5]. However, there are no available numerical tools for general systems allowing for the detection and continuation of sliding bifurcations. In what follows, taking as an example the dry friction oscillator (1), we will present a novel numerical technique to allow the



(a) Crossing sliding (b) Grazing sliding (c) Switching sliding

Figure 1: Bifurcation scenarios depicting the interaction of a trajectory with the boundary of the sliding region when (a) bifurcating limit cycle switches between F_1 and F_2 vector fields at the bifurcation point, (b) one of the vector fields (F_1 or F_2) has a point of tangency with Σ at the boundary of $\hat{\Sigma}$ and (c) the vector field switches between F_1 or F_2 to F_s at the bifurcation point.

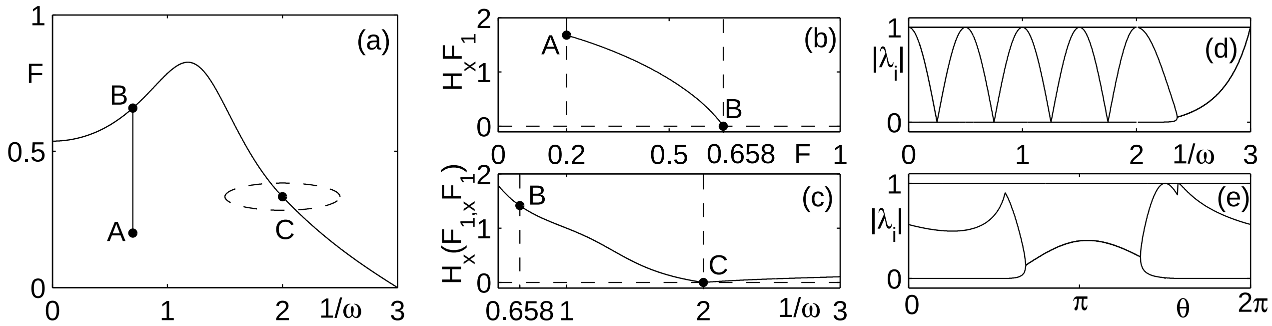


Figure 2: (a) A two-parameter bifurcation diagram. (b),(c) Analytical conditions for the detection of codimension-1 (b) and codimension-2 (c) sliding bifurcations under F and ω variations, respectively. (d),(e) The magnitude of the floquet multipliers λ_i along the codimension-1 branch (d) and along the ellipse (e) in Fig. 1(a).

detection and continuation of periodic orbits in systems with sliding, to follow branches of codimension-1 sliding bifurcations and to detect codimension-2 degenerate sliding bifurcations.

Briefly, the technique is based on the use of Newton's method to locate periodic orbits, for which the first variational equations are solved. Moreover, to account for the presence of switchings between different flows, local discontinuity mapping methods are used [8]. A pseudo-arclength strategy is used for the continuation of limit cycles in one or two parameters, which allows for continuation around fold points. Bifurcation detection is achieved through the set of analytical conditions for the occurrence of sliding bifurcations presented in [2, 5].

Fig. 2 shows some representative numerical results. Here, a periodic orbit located at the point 'A' in Fig. 2(a), characterised by crossing Σ twice per period outside of the sliding region, is continued in two-parameter space. By using the technique described above, a crossing-sliding bifurcation was successfully detected at $F = 0.658$ (see 'B' in Fig. 2(a)) when the analytical condition for its detection is found to be zero (see Fig. 2(b)).

The crossing-sliding bifurcation branch was then followed in two parameters (F and ω) while monitoring the analytical conditions for degenerate codimension-2 sliding bifurcations to occur. A codimension-2 bifurcation was then found numerically for the first time at the point 'C' in Fig. 2(a). (Fig. 2(c) shows the zero-crossing of the analyical condition characterising such bifurcation).

As expected from the analysis presented in [5] we were able to locate two additional codimension-1 bifurcations curves branching out from 'C' of which we continued the grazing-sliding bifurcation curve (see Fig. 2(a)). It was found that the numerics matches the analytical expectations as the bifurcationg limit cycle preserves its stability and period while crossing the codimension-1 bifurcation curves locally to the codimension-2 node. This was confirmed by computing Floquet multipliers of the periodic orbit along the elipse given by $(\frac{1}{\omega}, F) = (2 + 0.5\cos(\theta), \frac{1}{3} + 0.05\sin(\theta))$ around 'C' (see Fig. 2(a), (e)). Finally, we also present Floquet multipliers (see Fig. 2(d)) calculated along the codimension-1 curve in Fig. 2(a).

CONCLUSIONS

Novel numerical techniques were developed for the detection and continuation of codimension-1 sliding bifurcations of limit cycles and applied to investigate the dynamics of a representative dry-friction oscillator model. Branches of codimension-1 sliding bifurctaions were found and continued. Furthermore, a codimension-2 bifurcation node was found. The numerical results are found to match the analytical expectations.

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