PARADOXICAL BEHAVIOUR OF VIBRATING SYSTEMS CHALLENGING RAYLEIGH'S THEOREM

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Summary Rayleigh has proved a theorem that an additional constraint cannot reduce any of the natural frequencies of an elastic vibration system. Since linear free vibration and linear stability problems are analogous to each other mathematically, both lead to eigenvalue problems involving linear operators, the question arises, whether there exist a counterexample to Rayleigh's theorem, because to a similar theorem on the critical loads, there are counterexamples. In this paper, examples of vibrating systems will be shown that contradict Rayleigh's theorem. So the known form of Rayleigh's theorem might need correction.

PROBLEM DESCRIPTION

There is a widely-held opinion in engineering that the reinforcement of an elastic structure, either by an increase of the stiffness or by the provision of additional constraints increases both its critical loads and natural frequencies. This view, however, has not proved to be correct in all aspects.

The general validity of this opinion was contravened first in the case of the effect of stiffening and additional restraints on the buckling load [4]. The effect of additional restraint on the natural frequencies of a conservative vibration system was investigated by Rayleigh [2]. He proved a theorem that an additional constraint or an increase in the stiffness cannot reduce any of the natural frequencies. This theorem is generally accepted and incorporated in all the most important textbooks on the subject.

In this paper, it will be shown that Rayleigh's theorem on the natural frequency increasing effect of constraints or stiffening might need correction. Examples of vibrating systems will be presented in which, paradoxically, an additional restraint decreases the corresponding natural frequencies that contradicts Rayleigh's more than hundred-year-old theorem.

EFFECT OF ADDITIONAL RESTRAINT UPON THE ELASTIC CRITICAL LOAD

Let us assume (a) the system has a potential; (b) only bifurcation problems are considered so that the critical load factor means the load factor associated with the point of bifurcation; (c) the material of the structure is linearly elastic; (d) the displacements are small. Under these conditions, the following theorems hold [4].

THEOREM 1. *Additional restraint on any component of displacement, which is found in the eigenvalue problem of buckling of a structure, cannot decrease the minimum positive critical load factor.*

THEOREM 2. *Additional restraint on any component of displacement, which is not found in the eigenvalue problem of buckling of a structure, can decrease the minimum positive critical load factor.*

Example

To illustrate Theorem 2, consider the structure in Fig. 1, composed of weightless rigid members of length l and weightless elastic springs with rigidities c_1, c_2, c, loaded by a vertical force P at its upper joint, but mass m is not taken into consideration. The critical load of this structure is

$$P_{cr} = \frac{cl}{2}\left(1 + \frac{c_1}{c_2}\right). \quad (1)$$

Expression (1) shows that with an increase of the spring rigidity c_2, the critical load decreases.

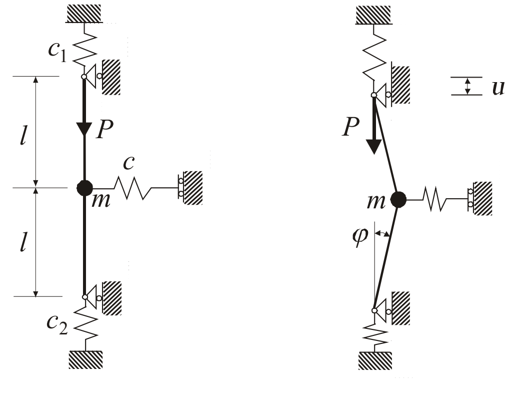

Fig. 1. The structure prior to and after displacements

EFFECT OF ADDITIONAL RESTRAINT UPON THE NATURAL FREQUENCIES

Consider a version of Rayleigh's theorem on constrained vibration as formulated by Renton [3]:
THEOREM 3. *"Suppose that the natural frequencies of a structure are* $\omega_1 \le \omega_2 \le \ldots \le \omega_r \le \ldots \le \omega_n$. *If one constraint is imposed on the structure, then the new natural frequencies of the structure, also arranged in order, will be such that the rth,* ω_r'*, will lie between the rth and the (r + 1)th original frequencies, or* $\omega_r \le \omega_r' \le \omega_{r+1}$*."*
Rayleigh's theorem on stiffening can be stated as follows.
THEOREM 4. *If the system stiffens, every natural frequency increases (or remains the same).*
Here, the term 'constraint' refers to an inhibition of one of the degrees of freedom of the structure; that is, the constraint removes that degree of freedom altogether. Stiffening does not mean simply an increase in the rigidity of one or more springs. According to Courant and Hilbert [1]: "Stiffening of the system means change to a system whose kinetic energy is the same but whose potential energy is greater for the same values of the coordinates." Thus, if a spring rigidity is increased, but as a consequence, the potential energy decreases, and so some of the natural frequencies decrease, then this fact does not contradict Theorem 4. However, if at least one natural frequency decreases by introducing an additional constraint, then Theorem 3 in the present form is not right.
These theorems are obviously valid if the potential energy means strain energy. However, there are vibration problems where static external forces are in the system, so these forces also contribute to the potential. Such a problem is, for instance, where an axial compression force P is on a vibrating beam, which combines two problems: (a) buckling of a beam with no vibration, (b) vibration of an unloaded beam. If the rth vibration and buckling modes are identical, then

$$\omega_r^2 = \omega_{r0}^2 \left(1 - \frac{P}{P_{r,cr}}\right) \tag{2}$$

where $P_{r,cr}$ is the rth critical force, ω_r is the rth natural frequency of the beam loaded by P, ω_{r0} is the rth natural frequency of the beam not loaded by P [5]. Thus, if additional restraint reduces the critical load then at the same time reduces the corresponding natural frequency.

Example
In Fig. 1, the structure with a mass m at its middle joint forms a two-degree-of-freedom vibration system. Let us suppose that $P < P_{cr}$ is a constant compression force, and $c_1 > c$. The critical force P_{cr} is given by (1).
(a) *An increase in the stiffness of an already existing restraint.* The system has two natural frequencies:

$$\omega_1^2 = \frac{c}{m}\left(1 - P\left[\frac{cl}{2}\left(1 + \frac{c_1}{c_2}\right)\right]^{-1}\right), \qquad \omega_2^2 = \frac{c_1 + c_2}{m}. \tag{3a, b}$$

Relationship (3a) has the same form as (2). If the spring rigidity c_2 is increased then ω_2 increases, but ω_1 decreases, that is, $\omega_1' < \omega_1 < \omega_2 < \omega_2'$, that seemingly contradicts Theorem 4. In fact, however, there is no contradiction because there are displacement coordinates for which the potential energy can decrease with an increase in c_2.
(b) *An addition of a constraint to the structure.* First, let $c_2 = 0$. This case means that we have a two-degree-of-freedom vibration system where the external force P does not play any role even it is present. In this case we have:

$$\omega_1^2 = \frac{c}{m}, \qquad \omega_2^2 = \frac{c_1}{m}. \tag{4a, b}$$

Then, let $c_2 = \infty$. The modified system obtained is a one-degree-of-freedom system where the vertical vibration mode ceases to exist. From (3a), we obtain for the modified system:

$$\omega_1'^2 = \frac{c}{m}\left(1 - \frac{2P}{cl}\right), \tag{5}$$

that is, $\omega_1' < \omega_1 < \omega_2$, that contradicts Theorem 3.

CONCLUSIONS

From the results presented here we can conclude that Rayleigh's theorem is valid if additional restraints have no effect on pre-vibration displacements, or there are no pre-vibration displacements, but additional restraints on pre-vibration displacements can decrease the natural frequencies. The latter case can happen if external static forces are in the system.

References

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