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DISCONTINUOUS SOLUTIONS OF THE BOUNDARY-LAYER EQUATIONS

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Summary In this presentation two examples of discontinuous solutions of the boundary-layer equations will be discussed. The first one represents an unsteady analogue of the well known two-dimensional laminar jet. We shall assume that the jet emerges from a narrow slit which was initially closed. The second example concerns the hypersonic boundary-layer flow on a delta wing in the regime of strong viscous-inviscid interaction.

It is already one hundred years since the time when Prandtl (1904) formulated the boundary-layer equations. Starting with his work it was always assumed that, due to the viscous nature of the boundary layers, the solution of the Prandtl equations should be sought in the class of continuous functions. Meanwhile, it can be easily seen that there are clear mathematical reasons for discontinuous solutions to exist. Moreover, discontinuous solutions not only form naturally in unsteady and/or three-dimensional flows, but under certain conditions represent the only possible solutions of the boundary-layer equations. In this paper we will give two examples of such flows.
The first one represents an unsteady analogue of the laminar jet studied by Schlichting as early as in 1933. In Schlichting's steady flow formulation the jet emerges from a narrow slit, as shown in Figure 1, and mixes with otherwise stagnant fluid to the right of the barrier OO'. As a result of the action of viscous stresses the fluid in the jet gradually loses its speed. Still the jet proves to be able to penetrate through the surrounding fluid over an infinite distance, with the velocity in the jet decaying gradually as $x \to \infty$. If the slit width is small, then the solution of the boundary-layer equations may be expressed in a simple analytical form

$$u = x^{-1/3} f'(\eta), \quad v = x^{-2/3}\Big(\frac{2}{3}\eta f' - \frac{1}{3} f\Big), \quad f(\eta) = \sqrt{6C}\,\frac{1-\exp\Big(-\sqrt{\frac{2}{3}C}\,\eta\Big)}{1-\exp\Big(\sqrt{\frac{2}{3}C}\,\eta\Big)}. \tag{1}$$

Here constant C characterises the strength of the jet; (x, y) are Cartesian coordinates with x measured along the jet axis and y in the normal direction; u, v are the velocity components in these coordinates and $\eta = y/x^{2/3}$ the similarity variable.

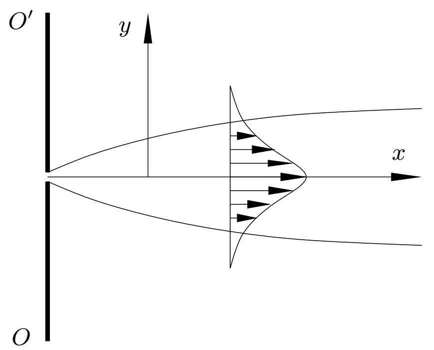

Figure 1. Problem layout

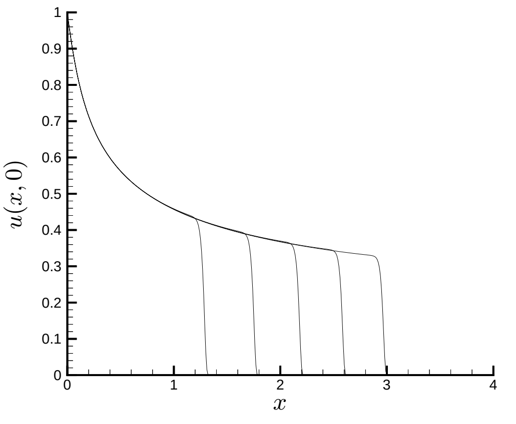

Figure 2. Velocity along the jet axis at $t = 4,\ 6,\ 8,\ 10$ and 12.

In our calculations we use unsteady boundary-layer equations. With t denoting time, the momentum and continuity equations are written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

The momentum equation is parabolic. It, apparently, does not allow for discontinuities in the y-direction to exist. However, the first two terms on the left hand side,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x},$$

represent a quasi-linear hyperbolic operator and, by analogy with one-dimensional gas dynamics, one can expect "shock waves" to form in the (t, x)-plane. For these to be captured a conservative scheme has to be used to calculate equations (2). The calculations were performed assuming that initially the fluid to the right of OO' was kept at rest. At $t = 0$ the slit opens, and a jet starts to form. To illustrate the results of the calculations we show in Figure 2 the distribution of the fluid velocity along the jet axis. The five curves correspond to increasing time $t = 4,\ 6,\ 8,\ 10$ and 12. It is easily seen that the jet has a well established front which propagates with finite speed in the x direction. Before the front the fluid remains at rest. Across the front the fluid velocity jumps to the value given by the first of formulae (1).

The thickness of the shocks in gas flows is known to be comparable with the molecular mean free path λ, which may be estimated as $\lambda = O(Re^{-1})$, where Re is the Reynolds number. The continuum description of fluid motion is not possible under these conditions. The "shocks" in boundary layers are significantly thicker. They extend in the longitudinal x-direction over a distance Δx comparable with the boundary-layer thickness $y = O(Re^{-1/2})$. Therefore the continuum description is well suited for analysing their internal structure, which is why we call them *pseudo-shocks*. Asymptotic analysis of the Navier-Stokes equations shows that the flow within a pseudo-shock may be treated as inviscid and quasi-steady provided that it is considered in the coordinate frame moving with the shock front. The governing equations may be written as

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0, \qquad \nabla^2\psi = -\omega, \tag{3}$$

where ψ is the stream function and ω the vorticity.

The results of a numerical solution of equations (3) with appropriate boundary conditions are presented in Figure 3 in the form of the streamline pattern. It is displayed in the coordinate frame moving with the shock front. One can observe that there is a stagnation point in the flow field. Applying the Bernoulli equation to the two streamlines that lie along the x-axis upstream and downstream of the stagnation point respectively, it may be found that the speed of propagation of the shock front $V_{\text{shock}} = \frac{1}{2}u_{\max}$. Here $u_{\max}$ is the maximum value of the longitudinal velocity in the jet immediately behind the shock. Figure 4 reproduces a photograph of a thermal jet made by Shlien & Boxman (1981). When comparing this photograph with the calculated streamline pattern it should be taken into account that the photograph was taken in the laboratory frame.

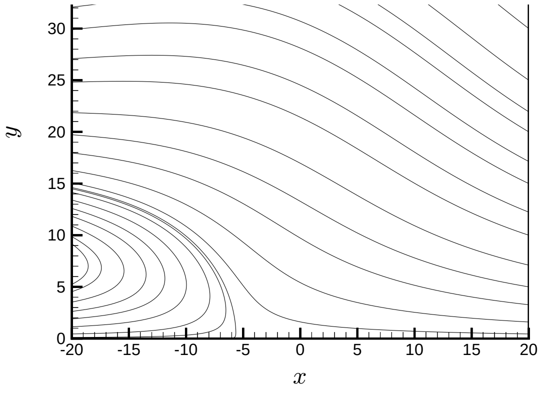

Figure 3. The streamline pattern.

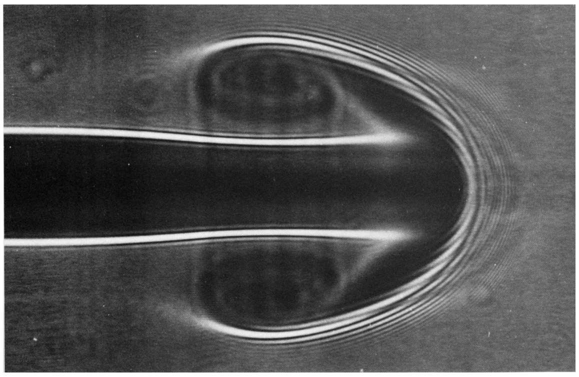

Figure 4. Visualisation of the jet head by Shlien & Boxman (1981).

In order to demonstrate that the pseudo-shocks are not uniquely attributed to unsteady two-dimensional boundary layers, we will also present the results of numerical analysis of the hypersonic flow past a delta wing. The calculations of this flow were performed under an assumption that the hypersonic viscous interaction parameter $\chi \gg 1$. In this case the boundary layer on the entire surface of the wing appears to be in strong interaction with the inviscid part of the flow. In these conditions the flow is governed by steady three-dimensional boundary-layer equations with self-induced pressure. We found that two pseudo-shocks are forming in the flow when the sweep angle of the wing is approaching a critical value. Contrary to earlier expectations (see, for example, Kozlova & Mikhailov, 1970) the shocks were found to lie off the plane of symmetry of the wing.

References

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