SOME DEGENERATED AND EXTENDED WAVE MODELS OF ELASTO- AND HYDRODYNAMICS WITH FINITE VELOCITY DISTURBANCE PROPAGATION

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Summary Hyperbolic degeneration of the problems for wave guide type systems with respect to transverse coordinate is considered. Some models are presented when parabolic models singularly degenerate into hyperbolic ones and, controversly, when hyperbolic models can be obtained by the extension of parabolic operator up to hyperbolic ones.

The problem of finite velocity disturbance propagation as a condition of the correctness of the statement of initial boundary value (IBV) problem for hyperbolic equations and its solvability have been considered by Hersh [1] presented some cases of nonexistence of the solutions of IBV problem with finite velocities. Correctly stated IBV problems for hyperbolic equations were investigated and the conditions guaranteeing a finite velocity were established. Later on the considerations of such a kind were developed in [2, 3, 4].

In general, degeneration of an original model in small parameter β leads to some simplified models of different kind. The necessary condition of finite velocity disturbance propagation is the hyperbolity of model, i. e. it should by governed by the hyperbolic system of partial differential equations. It should be noted that asymptotic degeneration can change the type of equations. That is why, only limiting hyperbolic models are of our interest from above point of view. For example, parabolic model degenerates into hyperbolic ones, hyperbolic elastodynamic model for layer degenerates into hyperbolic ones but not into classical parabolic model predicting infinite velocity of disturbance propagation.

We consider degeneration of the original IBV problem in R^n for a finite hyperbolic system of arbitrary order partial differential equations with respect to the transverse coordinate x^s of a hyperlayer $\Omega \subset R^n$ whose thickness is assumed to be much lesser than the others. It allows to construct power series expansions of desired functions and to reduce consequently the problem dimensionality. At that, the infinite system can be truncated by different manners allowing to obtain different approximations. In this connection three cases are possible: degeneration to models of hyperbolic, parabolic or mixed type. The question is what is correct ? From the point of view of finite velocity disturbance propagation only the case of hyperbolic to hyperbolic is correct and having the physical sense. However, it is reached by the price of degeneration of the original problem spectrum.

Our aim is to derive hyperbolic approximations degenerated with respect to coordinate x^s, i.e. to construct a mapping $R^n \to R^{n-1}$ satisfying the condition of limiting correctness to be of hyperbolic type [2]. Necessary and sufficient conditions are established to such a hyperbolic degeneration. The necessary condition is to obtain a closed system to determine new unknown functions. The close is necessary but not sufficient condition because a lot of closed systems can be obtained depending on the selection criteria but not all of them will be of hyperbolic type. The sufficient condition is announced for obtaining hyperbolic approximations: to keep in infinite systems all space-time differential operators up to given order. This is proved for the case R^3 considering the elastodynamic problem for the layer. As a result, 3-D problem for the layer is reduced to the extended 6 order hyperbolic operator for bending waves in plate which includes as particular cases the known Timoshenko-Mindlin (4 order) and Kirchhoff models.

Expanding field functions in power series with respect to a middle hyperlayer surface

$$u_i\left(t, x^2,..., x^{n-1}, x^n\right)=\sum_{k=1}^{\infty} u_{ik}\left(t, x^2,..., x^{s-1}, x^{s+1},..., x^n\right)\left(x^s\right)^k. \tag{1}$$

yields the degenerated problem to determine the series coefficients depending now only on n-1 coordinates..

Let us consider the elastodynamic problem for an infinite layer of the thickness $2h$ occupied the region $\Omega=\left\{(x_1, x_2, x_3) \in R^3 : x_1, x_2 \in (-\infty, \infty), \quad x_3 \in \left[-\xi/2, \ \xi/2\right]\right\}$ in a Cartesian coordinate system (x_1, x_2, x_3). The corresponding IBV-problem for the displacement vector ($u_1, u_2, u_3 = w$) is presented as follows: to find the functions $u_k = u_k(x_1, x_2, x_3, t)$ as the solutions of the hyperbolic equations

$$\nabla^2 u_k + (1+\lambda/G)\, \partial_k (\vec{\nabla}\cdot\vec{u}) = \partial_{tt}\, u_k, \quad k=1,2,3 \qquad \text{in} \quad \Omega\times[0, T], \ T>0 \tag{2}$$

satisfying the boundary conditions for the stress-tensor components

$$\sigma_{33}\big|_{x_3=\pm\xi/2} = q^{\pm}(x_1, x_2, t), \ \sigma_{3i}\big|_{x_3=\pm\xi/2} = p_i^{\pm}(x_1, x_2, t), \qquad i=1, 2 \tag{3}$$

and the initial conditions

$$u_k\big|_{t=0} = 0, \ \partial_t u_k\big|_{t=0} = 0, \qquad k=1,2,3. \tag{4}$$

It is assumed that $2h/l = \xi << 1$ (l is a horizontal length scale), λ and $G = const$.

The functions u_k are presented according to (1) in the form of power series in x_3

$$u_k\ (x_1, x_2, x_3, t)\ =\ \sum_{\nu=0}^{\infty} u_{k\nu}\ (x_1,\ x_2, t)\ x_3^{\nu}\ ,\qquad k=1,2,3\ . \tag{5}$$

In the case of symmetric deformation with respect to the middle surface we obtain from equations (2)-(4) (when $q_1,\ q_2=0$), as the second approximation – the new more exact two-mode hyperbolic equation for $e_0 = (u_0)_{i,\,i}$

$$\left(-\xi a_1'\nabla^2+\xi a_2'\frac{\partial^2}{\partial t^2}+\xi^3 b_1'\nabla^2\nabla^2-\xi^3 b_2'\nabla^2\frac{\partial^2}{\partial t^2}+\xi^3 b_3'\frac{\partial^4}{\partial t^4}\right)e_0=$$

$$=\left\{d_1' + \xi^2 d_2'\ \nabla^2 + \xi^2 d_3'\frac{\partial^2}{\partial\, t^2}\right\}\left(\frac{\partial}{\partial x_2}\frac{p_1-p_2}{2}+\frac{\partial}{\partial x_1}\frac{p_3-p_4}{2}\right). \tag{6}$$

The equation (6) includes as a particular case the one-mode hyperbolic approximation (generalized stress state): two the first terms in the left hand side and the first term in the right hand side.

In the case of asymmetric deformation, keeping all the operators up to seventh order inclusively, we obtain from (6)-(8) (when $p_1, p_2, p_3, p_4=0$) the new three-mode hyperbolic approximation which can be reduced to the following generalized resolving hyperbolic equation

$$\left\{\left[\left(\xi\frac{\partial^2}{\partial t^2}+\xi^2 a_1\nabla^2\nabla^2\right)_K-\xi^3 a_2\frac{\partial^2}{\partial t^2}\nabla^2+\xi^3 a_3\frac{\partial^4}{\partial t^4}\right]_{TM}\right. -$$

$$-\xi^5 b_1\nabla^2\nabla^2\nabla^2+\xi^5 b_2\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2-\xi^5 b_3\frac{\partial^4}{\partial t^4}\nabla^2+$$

$$\left.+\ \xi^5 b_4\frac{\partial^6}{\partial t^6}\right\}_{TMS}\quad w_0=\left\{\left[1-\xi^2 d_1\nabla^2+\xi^2 d_2\frac{\partial^2}{\partial t^2}\right]_{TM}+\xi^4 d_3\nabla^2\nabla^2\right. -$$

$$\left.-\ \xi^4 d_4\frac{\partial^2}{\partial\, t^2}\nabla^2+\xi^4 d_5\frac{\partial^4}{\partial t^4}\right\}_{TMS}\left(q^+-q^-\right) \tag{7}$$

In (7) the sixth order hyperbolic operator TMS includes the fourth order Timoshenko-Mindlin hyperbolic operator TM and the Kirchhoff parabolic operator K.

It can be noted that unlike phenomenologocal approaches, for realization of the purely analytical approach presented in this paper it is not necessary to attract some physical prerequisities to construct the hyperbolic models.

Another problem is considered on the basis of the algorithm above presented allowing to construct extended degenerated close to hyperbolic model for nonlinear water wave propagation in the fluid of variable depth. Starting from a fully nonlinear statement asymptotic approximations are developed when the dispertion parameter β is assumed to be small while there is no limitations on the value of the nonlinear parameter α unlike previous works. As a result, the extended nonlinear-dispersive evolution equations for solitary waves are derived which include all known models as particular cases. The evolution of soliton running-up to slope is investigated showing the soliton distortion and appearing oscillating trails.

In addition the singular degeneration in parameter of parabolic model into hyperbolic ones takes place in the cases when operator include as a kernel the hyperbolic operator of lower order. For example, Navier-Stokes equations, Burgers equation, equations of magnetoelasticity and magnetohydrodynamics when the magnetic Reynolds number tends to zero, and others.

Alternatively, some examples are presented when hyperbolic models can be obtained by the extension of parabolic operator up to hyperbolic once: kinetic theory of gases (Maxwell, 1876), diffusion equation (Davydov, 1935), Smoluchowski equation (Davies, 1954), relativistic body (Bento, 1985), sediment evolution (Selezov, 1985).

References

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