ON GENERATING CHAOTIC DYNAMICS IN NOLINEAR VIBRATING SYSTEMS

Elżbieta Tyrkiel
IPPT PAN, Department of Dynamics of Complex Systems, PL 00-049 Warszawa, Poland

Summary The paper is aimed at demonstrating the mechanism triggering chaotic phenomena in nonlinear dynamical systems, i.e. the formation of nonattracting invariant chaotic sets (chaotic saddles), which originates from the global bifurcations. Characteristic examples of the resulting chaotic system behaviors, such as chaotic transient motions, fractal basin boundaries and an unpredictability of the final outcome, are shown and discussed. Numerical study is carried out for two low dimensional but representative models of nonlinear dissipative oscillators driven externally by periodic force.

INTRODUCTION

A central problem in nonlinear dynamics is that of discovering how the qualitative dynamical properties of regular solutions change and evolve as the dynamical system is continuously changed. It is known that nonlinear systems typically possess more than one solution at given values of control parameters, and that two or more solutions may be locally stable. In periodically forced oscillators, typical nonlinear resonance responses reveal various types of sudden changes (bifurcations) of coexisting stable periodic orbits, resulting in a qualitatively new state when forcing parameters are varied. However, there are the existing unstable solutions, which, although physically unrealizable, play decisive roles in the organization of the phase space of a dynamical system, and lead to sudden changes in system behavior that are not caused by local bifurcations. These changes belong to the category of global bifurcations and are typically triggered by tangencies, and further transversal intersections of the invariant manifolds associated with the unstable orbits [4,5]. With the sweep of the forcing parameter, it results in creating an uncountable nonattracting invariant set, called a chaotic saddle [1,2]. Mathematically, chaotic saddle is a closed, bounded invariant set having a dense orbit (i.e. including a Cantor set). It means that it possesses a chaotic trajectory that never leaves the phase-space region containing the set, while almost every trajectory leaves the region after some transient time, possibly reaching a remote attractor. In other words, all initial conditions except for a set of Lebesque measure zero leave the region. This type of dynamics is referred to as a horseshoe-type one, as it is similar to the “stretching and contracting” action of the prototypical two-dimensional horseshoe map possessing a hyperbolic invariant set [6,7]. Existence of a chaotic saddle is the source of complexity of the system dynamics (even in simple one-dimensional systems), as it generates sensitivity to initial conditions, and this implies chaotic responses.
Theoretically, the chaotic trajectory initialized exactly at the point belonging to a chaotic saddle will stay on this set for ever, wandering to and fro, but this situation is neither physically nor numerically realizable, due to round-off and the exponential growth in errors. In practice, the orbit can spend a possibly long time in the vicinity of the chaotic saddle before it leaves, thus producing chaotic transient motion. In a large class of oscillators with at least one maximum of potential energy, in certain regions of control space chaotic transient may be characterized by multiple crossings of the potential barrier [4,5]. This makes the question of necessary conditions for the chaotic transient to occur to be an important point in the analysis of dynamical responses of engineering systems.

MODELS OF OSCILLATORS AND NUMERICAL TECHNIQUES

In the paper, characteristic examples of the chaotic system dynamics due to expanding chaotic saddles as a result of the sequence of global (homoclinic and heteroclinic) bifurcations, with the increase of the forcing parameter, are presented and discussed, making use of the two low-dimensional mathematical models of nonlinear, strictly dissipative oscillators driven externally by periodic force.
The first model is the twin-well potential system (Duffing’s oscillator):

$$\ddot{x}+h\dot{x}-\tfrac{1}{2}x+\tfrac{1}{2}x^3=F\sin\omega t, \qquad T=2\pi/\omega, \tag{1}$$

with the potential energy having two minima (stable equilibrium positions) at $x=\pm 1$, and one maximum (unstable equilibrium position, potential barrier) at $x=0$.
The second model is the plane pendulum system

$$\ddot{x}+h\dot{x}+\sin x=F\sin\omega t, \qquad T=2\pi/\omega, \tag{2}$$

with the potential energy having minima at $x=2n\pi$ and maxima at $x=(2n-1)\pi$, ($n=\pm\,0, 1, 2, 3....$). If the system is viewed in cylindrical space, it can be treated as a single-well potential system, with single stable (hanging) and single unstable (inverted, potential barrier) equilibrium positions.
For the sake of convenience, both equations are sought in a nondimensional form, with the time rescaled by linear eigenfrequencies of the undamped systems. With the fixed damping coefficient h (h = 0.1), the control parameters are the amplitude F and the frequency ω of the driving force.
The oscillators belong to archetypal representative models for the analysis of inherently nonlinear phenomena and are often used in engineering dynamics because they model a wide class of multidimensional systems whose dynamics can be captured by a single active mode (e.g. buckled beams, offshore structures, buildings in earthquake etc.). In both cases, we confine attention to the region of the principal resonance with respect to driving force ($\omega\approx 1$), i.e. the region

where the nonlinear resonance hysteresis occurs.
In the study we use ideas and numerical techniques of the nonlinear dynamics and chaos, which lead to clear interpretation of the system regular and chaotic properties in particular regions of the forcing parameters. The most valuable in studying the related phenomena is the concept of the *Poincaré return map* which consists in discrete time sampling of the periodic motion $x(t) = x(t+T),\ \dot{x}(t) = \dot{x}(t+T)$ in the phase space (with the sampling time equal to the known excitation period $T = 2\pi/\omega$), thus allowing to study the 3D flow governed by the ordinary differential equation by considering the associated 2D map. The idea of Poincaré map is successfully applied for the visualization of the *basins of attraction* of the stable solutions (attractors), i.e. the domains of all initial conditions in the Poincaré section plane whose sampled trajectories approach asymptotically the particular attractors. The geometrical representation of the stable and unstable invariant manifolds in the Poincaré section, in connection with the related basin-phase structure and the detected existence of a chaotic saddle, allows us to establish the appropriate critical thresholds of global bifurcations, which define the sequential build-up of chaotic system responses.
We detect and visualize the existing chaotic saddle by plotting numerically a chaotic trajectory that never leaves the saddle. Such numerically obtained trajectory is referred to as a *saddle straddle trajectory*, and the method capable of finding it is known as the PIM (Proper Interior Maximum) triple method [2].
The figures have been created using the *Dynamics* software [3].

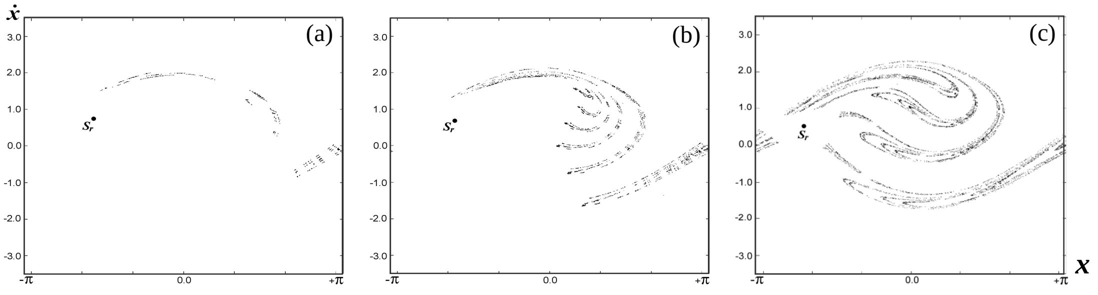

Fig. 1. Expansion of a chaotic saddle due to the Melnikov homoclinic bifurcation in a pendulum system (2), in the region of existence of a single attractor S_r (ω = 0.84); (a) F = 0.30; (b) F = 0.37; (c) F = 0.57.

CONCLUDING REMARKS

The phenomenon of formation of the nonattracting invariant chaotic sets (chaotic saddles) that occurs in nonlinear dissipative oscillators as the forcing parameter changes, and its crucial role in generating chaotic dynamics (sensitivity to initial conditions), is highlighted. It is shown that the phenomenon is strictly related to a sequence of the global (homoclinic and heteroclinic) bifurcations, i.e. transversal intersections of stable and unstable invariant manifolds of the particular existing unstable orbits. The lowest critical threshold of formation of a chaotic saddle is defined by the homoclinic bifurcation of the unstable orbit that corresponds to the potential barrier, and can be derived analytically (Melnikov criterion [7]). It is demonstrated that creation of a chaotic saddle manifests itself by the appearance of chaotic transient motions of an unpredictable, possibly long time duration. as well as that transient chaos typically appears at the level of control parameter values much lower than the one giving rise to a steady-state chaos (chaotic attractor). If two or more attractors with their basins of attraction coexist, chaotic saddle (embedded in a basin boundary) destroys its regular one-dimensional structure and generates fractal structure of the boundary, thus causing a loss of predictability of the final attractor reached. However, numerical evidence has been presented that, due to an existing chaotic saddle, chaotic transient motions are also generated in the case of a single attractor (Fig. 1).
The study allows establishing critical values of forcing parameters which are important in safe engineering, i.e. which define the domains of the safe (regular and predictable), relatively safe (possible chaotic, but confined to the inside of the potential well) and unsafe (chaotic with crossing the potential barrier and so dangerous for the structure) transient system motion.

References

[1] Lai, Y.C., Życzkowski, K., Grebogi, C.: Universal Behavior in the Parametric Evolution of Chaotic Saddles. *Physical Review E* **59**(3):5261-5265, 1999.

[2] Nusse, H.E., Yorke, J.A.: A Procedure for Finding Numerical Trajectories on Chaotic Saddles, *Physica **D*** **36**:137-156, 1989.

[3] Nusse, H.E., Yorke, J.A.: Dynamics: Numerical Explorations. Springer-Verlag NY 1998.

[4] Szemplińska-Stupnicka, W., Tyrkiel E.: Sequences of Global Bifurcations and the Related Outcomes after Crisis of the Resonant Attractor in a Nonlinear Oscillator. *Int. J. Bifurcation and Chaos* **11**(7):2437-2457, 1997.

[5] Szemplińska-Stupnicka W., Tyrkiel E.: The Global Bifurcations That Lead to Transient Tumbling Chaos in a Parametrically Driven Pendulum. *Int. J. Bifurcation and Chaos* **10**(9):2161-2175, 2000.

[6] Ott, E.: Chaos in Dynamical Systems. Cambridge University Press, Cambridge NY 1993.

[7] Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer-Verlag NY 1990.