EXTENDED NONLINEAR THEORY FOR TOPOGRAPHIC ROSSBY WAVES

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Summary The nonlinear-dispersive theory is proposed for topographic Rossby waves of small but finite amplitude for the case when external conditions are changed mainly along one direction, and wave is propagating along the orthogonal direction. The theory is based on the asymptotic procedure applied to hydrodynamic equations of frictionless, vertically homogeneous, incompressible, rotating fluid.

The equations of frictionless, vertically homogeneous, incompressible, rotating fluid can be reduced to the only one [1]:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\omega+\Omega}{h}\right)=0\,, \tag{1}$$

where x, y – are horizontal co-ordinates, t – time, (u, v) – corresponding components of horizontal velocity vector, h – ocean depth, ω - vorticity, Ω - Coriolis parameter. Let us choose Cartesian co-ordinates in such a way that Ω and h are variable mainly in y, and waves are propagating along x-axis. Using relations $\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$, $uh=-\frac{\partial\psi}{\partial y}$, $vh=\frac{\partial\psi}{\partial x}$ and considering nondimensional variables we can transform (1) to:

$$\left(h\frac{\partial}{\partial t}-\varepsilon\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}+\varepsilon\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{1}{h}\left[\mu\varepsilon\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial\psi}{\partial x}\right)+\varepsilon\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial\psi}{\partial y}\right)+\Omega\right]\right)=0\,, \tag{2}$$

where $\mu = (L_y/L_x)^2$ is the small parameter of topographic dispersion (L_y is the characteristic length scale in y-direction, L_x – characteristic wavelength), ε is the small parameter describing wave amplitude.
Let us consider long waves with small, but finite amplitude and put $\mu = \varepsilon$. If c is the speed of linear long wave (yet to be determined) we introduce the new variables [2] $\tilde{t}=t-\int_0^x\frac{d\xi}{c(\xi)}$, $x_i=\varepsilon^i x$ ($i\geq 1$). It follows that

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\tilde{t}}\,,\qquad \frac{\partial}{\partial x}=\sum_{i=1}^{\infty}\varepsilon^i\frac{\partial}{\partial x_i}-\frac{1}{c(x)}\frac{\partial}{\partial\tilde{t}}\,. \tag{3}$$

Next we assume that our Rossby wave field has the asymptotic expansion

$$\psi(X,y,\tilde{t})=\sum_{i=0}^{\infty}\varepsilon^i\psi_i(X,y,\tilde{t})\,. \tag{4}$$

Here X is a vector representing multiple scales (x_1, x_2, …). After substitution of (3), (4) into (2) and collecting terms of the same order of ε, we obtain the sequence of linear inhomogeneous boundary problems

$$L\frac{\partial\psi_i}{\partial\tilde{t}}=R_i\,, \tag{5}$$

$$\psi_i = 0 \text{ at } y = 0,l \qquad\text{or}\qquad \psi_i\to 0 \text{ at } y\to\pm\infty. \tag{6}$$

Here L is the linear operator determined by the expression $L\equiv\frac{\partial}{\partial y}\left[\frac{1}{h}\frac{\partial}{\partial y}\right]-\frac{1}{c}\frac{\partial}{\partial y}\left[\frac{\Omega}{h}\right]$. The lowest-order ($i$ = 0) boundary problem (5), (6) is homogeneous, variables can be separated: $\psi_0=A(X,\tilde{t})F(y,X)$, and $F(y,X)$ and $c(X)$ can be found from (5), (6) for any fixed Rossby wave mode. The compatibility conditions for the equation (5) for higher orders let us determine yet unknown slow-variable derivatives from A. For example, in the first order (i = 1) we obtain

$$R_1=C_{11}\frac{\partial A}{\partial x_1}+C_{12}\frac{\partial^3 A}{\partial t^3}+C_{13}A\frac{\partial A}{\partial t}+C_{14}A\,, \tag{7}$$

$$C_{11}=-F\frac{\partial}{\partial y}\left[\frac{\Omega}{h}\right],\ C_{12}=-\frac{F}{c^2h},\ C_{13}=\frac{F^2}{c^2h}\left(\frac{\partial^2}{\partial y^2}\left[\frac{\Omega}{h}\right]-\frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial}{\partial y}\left[\frac{\Omega}{h}\right]\right),\ C_{14}=\frac{\partial F}{\partial y}\frac{\partial}{\partial x}\left[\frac{\Omega}{h}\right]-\frac{\partial F}{\partial x}\frac{\partial}{\partial y}\left[\frac{\Omega}{h}\right] \tag{8}$$

$$\frac{\partial A}{\partial x_1}=-\beta\frac{\partial^3 A}{\partial t^3}-\alpha A\frac{\partial A}{\partial t}-qA \tag{9}$$

$$\psi_1 = \frac{\partial^2 A}{\partial t^2} T^{(d)}(y,X) + A^2 T^{(n)}(y,X) + \int A dt \ T^{(x)}(y,X) \,, \tag{10}$$

Here $T^{(d),(n),(x)}(y,X)$ – corrections to the mode $F(y,X)$, they are the solutions of

$$LT^{(d)} = -\beta C_{11} + C_{12} \,, \qquad LT^{(n)} = -\alpha C_{11} + C_{13} \,, \qquad LT^{(x)} = -q C_{11} + C_{14} \tag{11}$$

with boundary conditions similar to (6). Boundary-value problems (11) are solvable if the functions in the right-hand side are orthogonal to eigenfunction F of operator L. It gives us the expressions for α, β and q:

$$\alpha = \frac{\int FC_{13} dy}{\int FC_{11} dy} \,, \qquad \beta = \frac{\int FC_{12} dy}{\int FC_{11} dy} \,, \qquad q = \frac{\int FC_{14} dy}{\int FC_{11} dy} \,, \tag{12}$$

Now substituting $\partial A/\partial x_1$ (9) with coefficients (12) into series for $\partial A/\partial x$ (3), and keeping the terms up to current order i in ε we obtain the nonlinear evolution equation:

$$\frac{\partial A}{\partial x} + \varepsilon \left(\beta \frac{\partial^3 A}{\partial t^3} + \alpha A \frac{\partial A}{\partial t} + qA \right) = O(\varepsilon^2) \,, \tag{13}$$

which is the generalized Korteweg – de Vries equation with variable coefficients [1].
Continuing the procedure of expansion we obtain in the next order in ε:

$$R_2 = C_{21} \frac{\partial A}{\partial x_2} + C_{22} \frac{\partial^5 A}{\partial t^5} + C_{23} A^2 \frac{\partial A}{\partial t} + C_{24} A \frac{\partial^3 A}{\partial t^3} + C_{25} \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2} + C_{26} \frac{\partial^2 A}{\partial t^2} + C_{27} A^2 + C_{28} \int A dt + C_{29} \frac{\partial A}{\partial t} \int A dt \,, \tag{14}$$

$$\frac{\partial A}{\partial x_2} = -s_2 \frac{\partial^5 A}{\partial t^5} - s_3 A^2 \frac{\partial A}{\partial t} - s_4 A \frac{\partial^3 A}{\partial t^3} - s_5 \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2} - s_6 \frac{\partial^2 A}{\partial t^2} - s_7 A^2 - s_8 \int A dt - s_9 \frac{\partial A}{\partial t} \int A dt \,, \tag{15}$$

$$\frac{\partial \psi_2}{\partial t} = \frac{\partial^5 A}{\partial t^5} T_2 + A^2 \frac{\partial A}{\partial t} T_3 + A \frac{\partial^3 A}{\partial t^3} T_4 + \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2} T_5 + \frac{\partial^2 A}{\partial t^2} T_6 + A^2 T_7 + \int A dt \ T_8 + \frac{\partial A}{\partial t} \int A dt \ T_9 \,, \tag{16}$$

where second-order corrections T_k, k = 2…9, can be found from inhomogeneous boundary-value problems:

$$LT_k = -s_k C_{21} + C_{2k} \,, \ k = 2\ldots9, \tag{17}$$

with boundary conditions similar to (6). Compatibility conditions for problems (17) give the expressions for s_k, k = 2…9:

$$s_k = \frac{\int FC_{2k} dy}{\int FC_{21} dy} \ . \tag{18}$$

Substitution of $\partial A/\partial x_2$ (15) with coefficients (18) into series for $\partial A/\partial x$ (3) generates the equation:

$$\frac{\partial A}{\partial x} + \varepsilon \left(\beta \frac{\partial^3 A}{\partial t^3} + \alpha A \frac{\partial A}{\partial t} + qA \right) + \varepsilon^2 \left(s_2 \frac{\partial^5 A}{\partial t^5} + s_3 A^2 \frac{\partial A}{\partial t} + s_4 A \frac{\partial^3 A}{\partial t^3} + s_5 \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2} + s_6 \frac{\partial^2 A}{\partial t^2} - s_7 A^2 + s_8 \int A dt + s_9 \frac{\partial A}{\partial t} \int A dt \right) = O(\varepsilon^3) \tag{16}$$

As a result, temporal evolution and spatial transformation of Rossby wave field is described by nonlinear evolution equation (14) of second order in small parameters. This equation contains nonlinear, dispersive, cross nonlinear-dispersive terms, and the terms due to inhomogeneous medium (along the wave propagation axis). Variable coefficients α, β and q (12) and s_k (k = 2..9) of this equation are found in explicit form of integrals (over cross-propagation domain) depending on the wave modal structure, which is determined by a set of boundary-value problems. They are not presented here because they have quite complex form. In the case of homogeneous medium derived equation coincides with the famous second-order Korteweg – de Vries equation. The produced theory is valid for any Rossby wave mode. Because all the formulae derived are rather complicated we have used the symbolic computations for the verification.
The extended evolution equation derived here lets more precisely describe Rossby wave field, but there are some situations when the second order corrections are fundamentally important. Such a situation occurs when the coefficients of the first-order terms in the evolution equation vanish for some medium conditions. In this case nonlinear dynamics of the waves is determined by the next-order terms.

References

[1] Odulo A.B., Pelinovsky E.N. About nonlinear topographic Rosby waves *Oceanology* **18(1)**:16-19, 1978.
[2] Pelinovsky E.N., Fridman V.E., Engelbrecht Yu.K. Nonlinear evolution equations, Tallin, Valgus, 1984.