MAGNETOHYDRODYNAMIC MOTION OF TOROIDAL MAGNETIC EDDIES

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Summary The magnetohydrodynamic motion of toroidal magnetic eddies are studied using a new contour-dynamics formulation. The eddies are assumed to be axisymmetric with purely poloidal velocity and toroidal magnetic fields. Numerical simulation shows that the magnetic energy does not decay to zero against expectation although its lower bound determined by helicity is zero. Spherical heads which are approximated by an exact solution are formed to prevent the magnetic energy from vanishing.

INTRODUCTION

Magnetohydrodynamic (MHD) flows appear in various phenomena in astrophysical and engineering contexts. When dissipation effects are negligible, there exist three invariants of the dynamics: total energy, cross helicity and magnetic helicity. Magnetic helicity, which measures topological complexity of magnetic-field lines, bounds magnetic energy as

$$E_M \geq C|H|, \quad E_M = \frac{1}{2}\int B^2, \quad H = \int \boldsymbol{A}_M \cdot \boldsymbol{B}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}_M, \tag{1}$$

where C is a constant determined by the geometry of the region. Thus, magnetic energy cannot vanish if $H \neq 0$.
In this paper, we are concerned with the converse: does magnetic energy vanish when $H = 0$ so that the helicity bound for E_M is zero? In other words, we seek other mechanisms which control magnetic energy. We investigate this problem for toroidal magnetic eddies which have compact cross sections. The toroidal magnetic eddy can be regarded as a simple model of curved magnetic tubes which are observed in, for example, solar flares; we expect the results below are extended and applied to actual phenomena.

CONTOUR DYNAMICS

We consider an axisymmetric MHD flow whose velocity and magnetic fields are purely poloidal and toroidal, respectively: $\boldsymbol{u} = u_r\boldsymbol{e}_r + u_z\boldsymbol{e}_z, \boldsymbol{B} = B_\theta \boldsymbol{e}_\theta$; note that helicity vanishes for this choice. Both viscosity and magnetic diffusivity are assumed to be zero. The toroidal component of magnetic field is further assumed to vanish outside of a compact cross section $f(r,z,t) > 0$ and be proportional to the distance from the axis of symmetry in the inside: $B_\theta = \kappa r H\left[f(r,z,t)\right]$, $H\left[\cdot\right]$ being the Heaviside function and κ being a constant. The fluid is assumed to be at rest initially. Then the flow is irrotational except on the boundary or contour $f = 0$ where singular vorticity is created because of the Lorentz force:

$$\omega/r = \Omega(r,z,t)\delta\left[f(r,z,t)\right]|\boldsymbol{\nabla} f|. \tag{2}$$

The MHD equations reduce to the following contour dynamics

$$\frac{\partial\gamma}{\partial t} = \kappa^2 R\frac{\partial R}{\partial s}, \tag{3}$$

$$\frac{\partial R}{\partial t} = u_r(R,Z,t), \quad \frac{\partial Z}{\partial t} = u_z(R,Z,t), \tag{4}$$

$$u_r(r,z) = \frac{1}{r}\oint \gamma\left[\frac{\partial}{\partial z}G(r,z|R,Z)\right]\mathrm{d}s, \quad u_z(r,z) = -\frac{1}{r}\oint \gamma\left[\frac{\partial}{\partial r}G(r,z|R,Z)\right]\mathrm{d}s, \tag{5}$$

$$G(r,z|r',z') = \frac{1}{2\pi}(rr')^{1/2}\left[\left(\frac{2}{k}-k\right)K(k) - \frac{2}{k}E(k)\right], \quad k^2 = \frac{4rr'}{(r+r')^2+(z-z')^2}. \tag{6}$$

Here $(r,z) = (R(s,t), Z(s,t))$ is a parametric representation of the contour with s as the parameter, γ is related to the strength Ω of the vortex sheet through $\gamma = \Omega R\left[(\partial_s R)^2 + (\partial_s Z)^2\right]^{1/2}$, and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively. The formulation above can be easily extended to multiple contours $B_\theta = r\sum_{i=1}^{m}\Delta\kappa_i H[f_i(r,z,t)]$, which in principle can approximate any continuous toroidal magnetic field with compact support in L^2-norm.
The contour dynamics derived above is closely related with the motion of an interface between two fluids which have different values of density [1]. Under appropriate assumptions, the motion of interface is described by a similar set of equations; buoyancy is the driving force of the motion and creates singular vorticity on the interface. For the present case, the Lorentz force drives the motion.

RESULTS

We have carried out some numerical simulations using the equations of contour dynamics (3) and (4). Standard techniques for related problems [2–4] are employed. The integrals in (5) are evaluated by the trapezoidal formula for discrete points

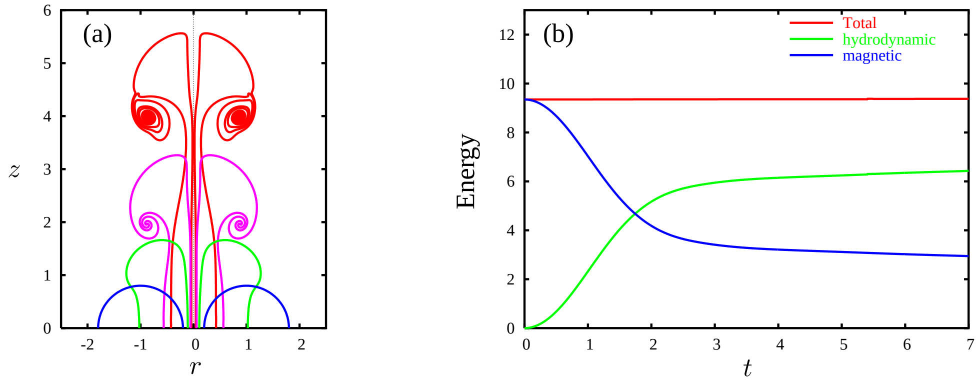

Figure 1. (a) Evolution of magnetic eddies. Cross section. $r_c = 0.8R_0$. $t = 0$ (blue), 2 (green), 4 (pink), 7 (red). (b) Evolution of energy: total energy (red), hydrodynamic energy (green), magnetic energy (blue).

which approximate the contours. They have divergent parts at $(r, z) = (R, Z)$, of which principal values are evaluated analytically [4]. As time proceeds in the numerical simulation, distance between neighbouring points becomes too large or too small depending on the positions to keep sufficient spatial resolution. To overcome this defect we reparametrize s at regular intervals.

The present method, however, suffers singularity which develops on the contours; it is essentially the same singularity that develops on a vortex sheet in non-MHD flow [2]. In order to track the global motion of the eddies for sufficient time, we regularize Green's function by introducing a smoothing parameter ϵ to the Biot-Savart law [3]: $4\pi \boldsymbol{A}_H = \int \boldsymbol{\omega}/x \;\rightarrow\; \int \boldsymbol{\omega}/\left(x^2+\epsilon^2\right)^{1/2}$.

Figure 1 shows an example of the numerical results. Figure 1(a) shows the evolution of the eddy; its cross sections by a plane $\theta = \mathrm{const.}$ are drawn for selected instants. The initial magnetic eddy is a torus whose cross section is a circle of radius $0.8R_0$ (R_0 is the radius of the torus). Only the upper half ($z > 0$) is shown since there is symmetry with respect to $z = 0$. The Lorentz force acts inward so that the eddy shrinks to z–axis as seen initially; one would expect the eddy eventually collapses to z–axis and correspondingly the magnetic energy vanishes. This is not prohibited since the helicity bound for the magnetic energy is zero in the present case. Two characteristic structures form as time proceeds: one is the spherical head which propagates steadily along z–axis and the other is the roll-up behind it. The number of turns in the roll-up increases with time; as a result fine structures emerge so that the dynamics cannot be pursued further.

Figure 1(b) shows time evolution of energy. The total energy is conserved with sufficient accuracy. The hydrodynamic energy, which is zero initially, starts to grow parabolically; after a stage of linear growth, the rate of growth becomes much smaller so that the magnetic energy appears to approach a constant. Correspondingly, the magnetic energy decreases monotonically and approaches a constant which is not zero although the helicity bound is zero.

This result on magnetic energy is closely related with formation of the spherical head. The head is well approximated by an exact solution which moves along z–axis with a constant speed $U_0 = (2/3)\,\kappa R_0$; it is represented in terms of Stokes' streamfunction as

$$\Psi(r, z_*) \;=\; \begin{cases} 0 & \sigma < R_0, \\ (1/2)\,U_0 r^2 \left[1 - (R_0/\sigma)^3\right] & \sigma > R_0, \end{cases} \tag{7}$$

where $(r, z_*) = (r, z - U_0 t)$ is a moving frame and $\sigma^2 = r^2 + {z_*}^2$. To summarize, the magnetic energy decays as the eddy shrinks to z–axis; however, it cannot decay to zero since a part of the eddy forms a spherical head close to an exact solution which maintains a finite amount of magnetic energy.

The flow field of the exact solution we have found is that of Hill's spherical vortex outside the sphere and is identically zero inside it; the gap between the two sides is compensated by the magnetic field and the vortex sheet. In fact there is a family of exact solutions which includes (7) and Hill's spherical vortex as members. Numerical test of the exact solution (7) shows that a spike like that observed for Hill's spherical vortex [4, 5] develops on the rear side while the front side appears to be stable. In addition to the formation of spike, singular vortex sheet is subject to the Kelvin-Helmholtz instability. Stability analysis in progress would clarify these two mechanisms of instability. Direct numerical simulation using the MHD equations, which is also in progress, will reveal the dynamics in more general cases including the case of non-vanishing helicity.

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