EVOLUTION OF PACKETS OF SURFACE GRAVITY WAVES OVER SMOOTH TOPOGRAPHY

E.S. Benilov, J.D. Flanagan, C.P. Howlin
University of Limerick, Ireland

INTRODUCTION

The dynamics of surface gravity waves over bottom topography is one of the classical problems of fluid mechanics. It has been thoroughly studied in various formulations, including linear/weakly-nonlinear monochromatic waves and shallow-water solitons (see Mei 1983 and references therein). However, an important particular case has been overlooked, as there seems to be virtually no results on shoaling of wave packets – which omission seems even stranger, if one recalls the significance of this problem for oceanography. The only exception is the paper by Barnes & Peregrine (1995), who showed that the behaviour of a shoaling packet is quite different to that of a monochromatic wave. It turned out that the amplitude of the former is much lower than that of the latter, as the packet tends to spread out – and this effect is more marked for slowly-varying topography. However, no quantitative results were obtained, which would link the parameters of the packet to the bottom topography over which it travels.

The present paper addresses this omission.

SURFACE WAVES OVER SMOOTH TOPOGRAPHY

Consider surface gravity waves in a channel with an uneven bottom. We shall assume that both topography and waves are one-dimensional, i.e. the depth H of the basin and the elevation η of the surface depend on a single horizontal coordinate, x (η also depends on the time t). We shall further assume that all our variables are non-dimensionalized using the mean depth H_0 and the acceleration due to gravity, g.

In this paper, we are concerned with weakly nonlinear, quasi-monochromatic waves (packets), which can be characterized by a frequency ω. If the bottom was flat, ω would correspond to a certain value of the wavenumber k, determined by the dispersion relation of surface gravity waves,

$$\omega^2 = k \tanh kH. \tag{1}$$

If H depends on x (uneven bottom), (1) can still be used as a means of determining k – which, however, will now depend on x (i.e. the frequency of the wave is fixed, while its wavenumber changes as it propagates over topography, see Djordjevic & Redecopp 1978). To justify the use of (1) for an uneven bottom, $H(x)$ should be a slowly-changing function of x, i.e. its spatial scale should be much larger than the wavelength $2\pi k^{-1}$.

Under these assumptions, the packet can be represented by

$$\eta(x,t) = \mathrm{Re}\left\{ A(x,t)\, \exp\left[i \int k(x)\, dx - i\omega t \right] \right\},$$

where $A(x,t)$ is a slowly-varying function, governed by a modified nonlinear Schrödinger (MNS) equation (Djordjevic & Redecopp 1978):

$$i\left(\frac{\partial A}{\partial x} + \frac{1}{c_g}\frac{\partial A}{\partial t} + \mu A\right) + \alpha \frac{\partial^2 A}{\partial t^2} + \beta A\,|A|^2 = 0. \tag{2}$$

In this equation, c_g is the group velocity of surface gravity waves and

$$\alpha = \frac{1}{2\omega c_g}\left(\frac{2\omega H \tanh kH}{c_g} + 1 - \frac{H}{c_g^2}\right), \qquad \beta = \frac{1}{2\omega^3 c_g}\left\{3k^4 + 2\omega^4 k^2 - \omega^8 - \frac{\left[2\omega k - c_g\left(\omega^4 - k^2\right)\right]^2}{H - c_g^2}\right\},$$

$$\mu = \frac{\left(1 - \tanh^2 kH\right)\left(1 - kH \tanh kH\right)}{\tanh kH + kH\left(1 - \tanh^2 kH\right)} \frac{d\,(kH)}{dx}.$$

Observe also that the MNS equation (2) is written in the form, where the spatial coordinate x is the "evolutionary" variable (which role is usually played by t).

ASYMPTOTIC ANALYSIS

If the bottom is flat, the coefficients of (2) become constant and, provided that $\alpha\beta > 0$, (2) admits an exact solution describing steadily propagating envelop soliton,

$$A = \sqrt{\frac{2\alpha}{\beta}}\lambda\, \mathrm{sech}\, \lambda\tau\, \exp\left(\frac{iv\tau}{2\alpha} + \frac{4i\lambda^2 x}{4 + v^2}\right), \tag{3}$$

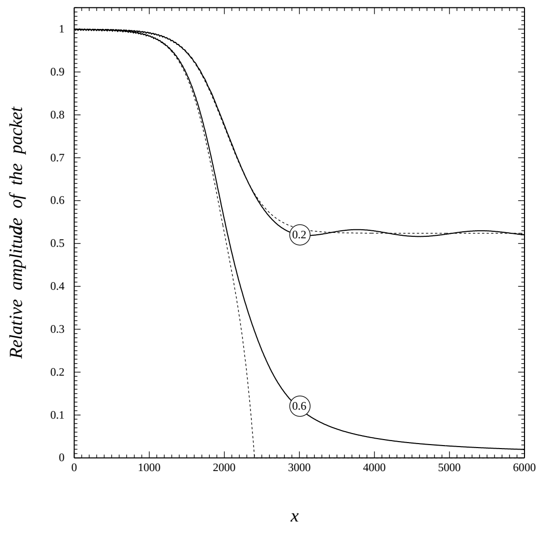

Figure 1. The evolution of wave packet over topography (5) (the curves are marked with the corresponding values of ΔH). The numerical solution of equation (2) is shown in solid line, the asymptotic solution (4) is shown in dotted line.

where $\tau = t - \left(c_g^{-1} + v\right) x$, and λ and v are arbitrary parameters. Observe that the amplitude of the wave packet is proportional to λ, whereas λ^{-1} is the width of the packet. Finally, v characterizes the packet's translation speed.

If the depth of the channel varies with x, but the horizontal scale of this variability is larger than the width of the packet, we can still assume the solution of (2) to be close to (3), but the parameters of (3), λ and v, should depend now on x. A straightforward asymptotic procedure results in the following "conservation law" for λ:

$$\frac{4\alpha\lambda}{\beta}\left(\tanh kH + kH \,\mathrm{sech}^2\, kH\right) = \mathrm{const}\,. \tag{4}$$

EXAMPLES AND DISCUSSION

We shall illustrate the use of (4) by the following example. Let the depth of the channel be

$$H(x) = 1 - \Delta H\ \tanh\left[0.002\ (x - 2000)\right], \tag{5}$$

which describes a step-like depth variation, of amplitude ΔH and width $(0.002)^{-1}$, located at $x = 2000$. The initial values of the packet are $k = 2$, $\lambda = 0.1$ at $x = 0$. First, we shall consider a small depth variation, $\Delta H = 0.2$, and calculate the evolution of the packet's amplitude using formula (4) – see Fig. 1. Remarkably – unlike linear waves or shallow-water solitons (see Mei 1983) – the amplitude of the shoaling packet decreases! This tendency was observed in all examples considered.

We have also present the results for a stronger depth variation, $\Delta H = 0.6$ (see Fig. 1), with all other parameters being the same as before. In this case, the asymptotic formula (4) predicts that, at $x \approx 2400$, the packet's amplitude vanishes and its width becomes infinite – which essentially means that the packet disintegrates.

To understand why this occurs, note that, at $kH = 1.363$, the nonlinearity coefficient β of the MNS equation vanishes. Note also that, as follows from (4), λ must vanish together with β. Hence, if $H(x)$ becomes sufficiently small, kH reaches the critical value, and the amplitude of the packet vanishes. Beyond this point, $\alpha\beta < 0$ – and the packet may no longer exist as a coherent solitary wave. Not dwelling on the details, we remark that a simple formula can be derived, relating the critical depth to the frequency of the packet: $H_{cr} \approx 1.195\,\omega^{-2}$. Thus if $H(x) > H_{cr}$, the packet passes over the depth variation as a coherent solitary wave – and disintegrates otherwise.

To verify the above conclusions, the exact MNS equation (2) has been simulated numerically using the psedospectral method. The results demonstrated that the agreement between the asymptotic and numerical solution is quite good (see Fig. 1). Finally, note that, strictly speaking, the asymptotic equation (4) is not applicable near the critical point. We shall not discuss this question in further detail, but refer the reader to the paper by Malomed & Shrira (1991).

References

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