TOPOLOGICAL OPTIMIZATION FOR CONTACT PROBLEMS

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Summary The problem of topology optimization is considered elastic contact problems. The formulae for the first term of asymptotics for energy functionals are derived and verified numerically. The topological differentiability of solutions to variational inequalities is established using the so-called *outer asymptotic expansion* for solutions of contact problems in elasticity with respect to singular perturbation of geometrical domain by nucleation of small holes.

Introduction. In the engineering literature there are many results concerning the shape optimization of contact problems in elasticity. The boundary variations technique for such problems is described in [4] in the framework of nonsmooth analysis combined with the speed method. Nonsmooth analysis is necessary since the shape differentiability of solutions to contact problems is obtained only in the framework of Hadamard differentiability of metric projections onto polyhedric sets in the appropriate Sobolev spaces. However, to our best knowledge, there is no general method for simultaneous shape and topology optimization [6] of contact problems. The main difficulty in analysis of contact problems is associated with the nonlinearity of the *non-penetration condition* over the contact zone which makes the boundary value problem nonsmooth. In the paper we propose a method for numerical evaluation of topological derivatives for such problems.
The main idea we use to derive the topological derivatives for contact problems is the modification of the energy functional by an appropriate correction term and subsequent minimization of the resulting energy functional over the cone of admissible displacements. Such an approach leads to the *outer* approximations of solutions to variational inequalities. The following step in asymptotic approximation of solutions is an application of self adjoint extensions of elliptic operators [3] with asymptotic point conditions for the contact problems which results in the asymptotically exact approximations. We point out that in the framework of self adjoint extensions studied in [3] for linear problems, the approximate solutions with asymptotic point conditions at the center of the hole $\mathcal{O}$ are critical points of the appropriate energy functional. We restrict ourselves to the case of outer approximations of solutions. Such an approach is justified, by applications to numerical methods of topology optimization. For linear problems, outer approximations are used e.g., in [1] for derivation of topological derivatives for isotropic elasticity. However, the complete asymptotic analysis necessary to justify the derivation of topological derivatives for general linear boundary value problems is performed in [2].

Contact problem in elasticity. The asymptotic analysis of 2D contact problems is performed in the framework of linear elasticity. Such a contact problem in the domain Ωreads: Find $\mathbf{u} = \mathbf{u}(\Omega) = (u_1, u_2)$ and $\sigma = (\sigma)_{ij}$, $i, j = 1, 2$, such that

$$-\mathbf{div}\, \sigma = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad C\sigma - \epsilon(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = 0 \quad \text{on } \Gamma_0\,,$$
$$\mathbf{u}\nu \geq 0, \quad \sigma_\nu \leq 0, \qquad \sigma_\nu \mathbf{u}\nu = 0 \qquad \sigma_\tau = 0 \quad \text{on } \Gamma_c\,.$$

The compliance functional for the contact problem is given by

$$J(\Omega) = \frac{1}{2}\int_\Omega \sigma(\mathbf{u}) : \epsilon(\mathbf{u})\, dx - \int_\Omega \mathbf{f}\mathbf{u}\, dx = -\frac{1}{2}\int_\Omega \mathbf{f}\mathbf{u}\, dx$$

where $\Omega \subset \mathbb{R}^2$ is a given domain with the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_c$. We obtain the expansion of this functional

$$J(\Omega_\rho) = J(\Omega) + \rho^2 \mathcal{T}_\Omega(\mathcal{O}) + o(\rho^2)$$

for $\rho^2 \to 0$, where $\Omega_\rho = \Omega \setminus \overline{B_\rho(\mathcal{O})}$, $B_\rho(\mathcal{O}) = \{\|\mathbf{x}\| < \rho\}$, and $\mathcal{T}_\Omega(\mathcal{O})$ is the topological derivative of $J(\Omega)$ at Ω for the nucleation of a small hole $B_\rho(\mathcal{O})$ with the center $\mathcal{O} \in \Omega$. To this end the contact problem in Ω_ρ is considered: Find $\mathbf{u} = \mathbf{u}(\Omega_\rho) = (u_1, u_2)$ and $\sigma = (\sigma)_{ij}$, $i, j = 1, 2$, such that

$$-\mathbf{div}\, \sigma = \mathbf{f} \quad \text{in} \quad \Omega_\rho\,, \quad C\sigma - \epsilon(\mathbf{u}) = 0 \quad \text{in} \quad \Omega_\rho\,, \quad \mathbf{u} = 0 \quad \text{on } \Gamma_0\,, \quad \sigma\nu = 0 \quad \text{on } \Gamma_\rho\,, \tag{1}$$
$$\mathbf{u}\nu \geq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu \mathbf{u}\nu = 0 \qquad \sigma_\tau = 0 \quad \text{on } \Gamma_c\,. \tag{2}$$

We show that the solution to (1)–(2), denoted $\mathbf{u}_\rho$, is conically differentiable with respect to the small parameter $\rho^2 \to 0$, i.e.

$$\mathbf{u}_\rho = \mathbf{u} + \rho^2 \mathbf{q} + o(\rho^2) \quad \text{in} \quad \Omega \setminus \overline{B(\rho)}$$

for small ρ^2, where $\mathbf{u}$ is a solution to the problem in Ω. The conical differential $\mathbf{q}$ is given by the unique solution of the auxiliary variational inequality: Find $\mathbf{q} \in S(\mathbf{u})$ such that

$$\int_\Omega \sigma(\mathbf{q}) : \epsilon(\mathbf{v} - \mathbf{q})\, dx + b(\mathbf{q}, \mathbf{v} - \mathbf{q}) \geq 0$$

for all $\mathbf{v} \in S(\mathbf{u})$, where $b(\cdot,\cdot)$ is a bilinear form and $S(\mathbf{u})$ is a cone, associated with the initial contact problem in Ω. Our results are based on a conical differentiability of solutions to variational inequalities defined over polyhedral convex sets [4]. These results are combined with an appropriate approximation procedure for the energy functionals, which leads to the bilinear form $b(\cdot,\cdot)$.

Approximation of energy. We determine the modified bilinear form as a sum of two terms, as it is for the energy functional, the first term $a(\mathbf{v},\mathbf{v})$ defines the elastic energy in the domain Ω, the second term $b(\mathbf{v},\mathbf{v})$ is a correction term. The correction term is quite complicated to evaluate, and we do not provide its explicit form. The values of the symmetric bilinear form $a(\rho;\cdot,\cdot)$ in Ω_ρ are given by the expression

$$a(\rho;\mathbf{v},\mathbf{v}) = a(\mathbf{u},\mathbf{u}) + \rho^2 b(\mathbf{v},\mathbf{v}) \ . \tag{3}$$

The derivative $b(\mathbf{v},\mathbf{v})$ of the bilinear form $a(\rho;\mathbf{v},\mathbf{v})$ with respect to ρ^2 at $\rho = 0+$ is given by the expression

$$b(\mathbf{v},\mathbf{v}) = -2\pi e_{\mathbf{v}}(0) - \frac{2\pi\mu}{\lambda+3\mu}\left(\sigma_{II}\delta_1 - \sigma_{12}\delta_2\right) \ , \tag{4}$$

where all the quantities are evaluated for the displacement field $\mathbf{v}$. The form of the first term in (4), new to our best knowledge, is given in terms of line integrals over the circle Γ_R. This circle of fixed radius surrounds the hole $B(\rho)$.

$$\begin{aligned} 2\pi e_{\mathbf{v}}(0) = {} & \frac{\pi(\lambda+\mu)}{\pi^2 R^6}\left(\int_{\Gamma_R} (v_1x_1 + v_2x_2)\, ds\right)^2 + \\ & + \frac{\mu}{\pi^2R^6}\left(\int_{\Gamma_R}\left((1-9k)(v_1x_1 - v_2x_2) + \frac{12k}{R^2}(v_1x_1^3 - v_2x_2^3)\right]ds\right)^2 + \\ & + \frac{\mu}{\pi^2R^6}\left(\int_{\Gamma_R}\left[(1+9k)(v_1x_2 + v_2x_1) - \frac{12k}{R^2}(v_1x_2^3 + v_2x_1^3)\right]ds\right)^2 \ , \end{aligned} \tag{5}$$

The values of $\sigma_{II}, \sigma_{12}, \delta_1, \delta_2$ are calculated by means of similar integrals and $k = (\lambda+\mu)/(\lambda+3\mu)$. Such expressions lead to very efficient numerical methods for the problem under consideration. Numerical results confirm this conclusion.

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