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GENERALIZED STRESS CONCENTRATION FACTORS

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Summary While the traditional stress concentration factor for a given loading is the ratio between the maximal stress in a body and the stress evaluated using simplified geometry, we regard the stress concentration factor as the ratio between the maximum of a stress component over the body, and the maximum value of the applied force fields. Then, for the given loading, we consider an optimal stress distribution which is a stress tensor field together with additional volume force density that will equilibrate the external loading and will result in the smallest stress concentration factor. Finally, the generalized stress concentration factor K is defined as the maximum of all optimal stress concentration factors for all external loading fields. The generalized stress concentration factor is clearly a geometric property of the body. It is shown that K is equal to the norm of the mapping that extends Sobolev functions defined in the interior of the body to its closure.

BASIC DEFINITIONS

Traditional stress concentration factors (see for example Peterson [2]) specify the ratio between the maximal value of a stress component in a body and the maximum value of that component for simplified, idealized geometry. The stress concentration factors are evaluated using analytical, numerical and experimental methods for given loadings and material properties. It is noted that the nominal stress calculated for the simplified geometry may be regarded as the boundary traction at a large distance away from the nontrivial geometry. For example, for a finite plate containing a hole, the nominal stress may be regarded as the boundary traction on the edge of the plate. This suggests that the stress concentration factor be represented by the ratio

$$K_F = \frac{\sup_{x,i,k}\{|\sigma_{ik}(x)|\}}{\sup_{i,x,y}\{|b_i(x)|, |t_i(y)|\}}, \quad x \in \operatorname{Int} B,\ y \in \partial B,$$

where b_i and t_i are the body force and surface force distributions associated with the given loading F. Thus, the distribution of the body force is also considered in the comparison. It is assumed throughout that the body is a compact 3-dimensional submanifold of $\mathbb{R}^3$ having a differentiable boundary.

In the last expression, the maximum over i in the denominator (e.g., $\max_i\{|b_i(x)|\}$) and the maximum over i, k in the numerator serve as norms on $\mathbb{R}^3$ and on the space $L(\mathbb{R}^3, \mathbb{R}^3)$ of linear mappings defined on $\mathbb{R}^3$. These may be replaced by other norms and we will use $|b(x)|$ and $|t(x)|$ to denote the norms of the values at $x \in B$ of the body force vector field b and surface force vector field t associated with the given loading F. We denote by $|\sigma(x)|$ the norm of the the value of the stress at x. Thus, the stress concentration factor may be written as

$$K_F = \frac{\sup_x\{|\sigma(x)|\}}{\sup_{x,y}\{|b(x)|, |t(y)|\}}, \quad x \in \operatorname{Int} B,\ y \in \partial B.$$

Here and in the following, we ignore high values of the stresses and body forces if they are restricted to subsets of the body of zero volume. Similarly, we ignore high values of the surface force if it is restricted to subsets of the boundary of zero area. Thus, the suprema over x and y are in effect essential suprema.

The concept of stress concentration is developed below in a number of steps. Firstly, we consider for the given external loading F, the collection Σ_F of all stress fields that are in equilibrium with F. We denote by $K^{F,\text{optimal}}$ the smallest stress concentration factor. This may be conceived as a process of structural optimization. Thus,

$$K^{F,\text{optimal}} = \frac{\inf_{\sigma\in\Sigma_F}\left\{\sup_x\{|\sigma(x)|\}\right\}}{\sup_{x,y}\{|b(x)|, |t(y)|\}}.$$

In the abstract notation σ, we consider not just stress tensor fields σ_{ik} but include self forces—force volume densities σ_i. The self forces may be thought of as additional body forces that one may apply in order to reduce the stresses or as "3-dimensional elastic foundations". Thus, this generalized stress field σ has 12 components and the norm of its value at any point should be modified accordingly (e.g., $|\sigma(x)| = \sup_{i,k,l}\{|\sigma_i(x)|, |\sigma_{kl}(x)|\}$). With the self forces, the equations of equilibrium assume the form

$$\sigma_{ik,k} + b_i = \sigma_i, \quad \text{in } \operatorname{Int} B.$$

Alternatively, the equivalent principle of virtual work is written as

$$\int_{\operatorname{Int} B} b_i w_i \,\mathrm{d}V + \int_{\partial B} t_i w_i \,\mathrm{d}A = \int_{\operatorname{Int} B} \sigma_i w_i \,\mathrm{d}V + \int_{\operatorname{Int} B} \sigma_{ik} w_{i,k} \mathrm{d}V.$$

Noting that one usually does not know the exact nature of the loading in advance, we allow the force distribution to vary and consider the worst case, i.e.,

$$K = \sup_F\left\{K^{F,\text{optimal}}\right\} = \sup_F\left\{\frac{\inf_{\sigma\in\Sigma_F}\left\{\sup_x\{|\sigma(x)|\}\right\}}{\sup_{x,y}\{|b(x)|, |t(y)|\}}\right\}.$$

We will refer to K as the *generalized stress concentration factor*. Clearly, the generalized stress concentration factor is a pure geometric property of the body.

THE RESULT: $K = \|\iota\|$

We prove that the generalized stress concentration factor K is equal to the norm of a mapping ι that takes a function defined in the interior $\operatorname{Int} B$ of the body and extends it to the closed body B (see [3]). Specifically, let $L^1_1(\operatorname{Int} B, \mathbb{R}^3)$ denote the Sobolev space of vector fields over the interior of the body with integrable components and integrable derivatives of the components. This space is equipped with the Sobolev norm

$$\|u\| = \int\limits_{\operatorname{Int} B} |u|\,\mathrm{d}V + \int\limits_{\operatorname{Int} B} |\nabla u|\,\mathrm{d}V.$$

Then, one of the basic properties of Sobolev functions (see [1]) implies that each Sobolev vector field $u \in L^1_1(\operatorname{Int} B, \mathbb{R}^3)$ may be extended to B in such a way that the restriction of the extension to the boundary, its trace $\hat{u}$, is integrable over the boundary. We use $L^{1,\mu}(B, \mathbb{R}^3)$ to denote the space of integrable vector fields over the body whose restrictions to the boundary are integrable over the boundary. For $w \in L^{1,\mu}(B, \mathbb{R}^3)$ we use the norm

$$\|w\|^{L^{1,\mu}} = \int\limits_{\operatorname{Int} B} |w|\,\mathrm{d}V + \int\limits_{\partial B} |w|\,\mathrm{d}A.$$

Thus, the norm of the extension mapping $\iota \colon L^1_1(\operatorname{Int} B, \mathbb{R}^3) \to L^{1,\mu}(B, \mathbb{R}^3)$ is given by

$$\|\iota\| = \sup_u \frac{\|\iota(u)\|}{\|u\|} = \sup_{u \in L^1_1(\operatorname{Int} B, \mathbb{R}^3)} \frac{\int\limits_{\operatorname{Int} B} |u|\,\mathrm{d}V + \int\limits_{\partial B} |\hat{u}|\,\mathrm{d}A}{\int\limits_{\operatorname{Int} B} |u|\,\mathrm{d}V + \int\limits_{\operatorname{Int} B} |\nabla u|\,\mathrm{d}V},$$

which is clearly a geometric property of the body.

OUTLINE OF THE PROOF

The proof (see [3] for the details) uses standard results of analysis. The various function spaces and mappings involved are presented in the diagram below. Specifically, one uses the duality of the L^1 and L^∞ spaces and the fact that $\|\iota\| = \|\iota^*\|$, where ι^* denotes the mapping dual to the extension ι. Referring to elements of $L^1(\operatorname{Int} B, \mathbb{R}^{12})$ as local virtual velocities, the mapping $j \colon L^1_1(\operatorname{Int} B, \mathbb{R}^3) \to L^1(\operatorname{Int} B, \mathbb{R}^{12})$ is defined by $j(u) = (u, \nabla u)$ and one can easily see that it is a norm-preserving injection. This implies that every element $S \in L^1_1(\operatorname{Int} B, \mathbb{R}^3)^*$ is of the form $S = j^*(\sigma)$ for some stress field σ. In addition, the dual norm of S may be calculated using

$$\|S\|^{L^1_1} = \inf_{S = j^*(\sigma)} \left\{ \|\sigma\|^{L^\infty} \right\}.$$

Finally, equilibrium or equivalently, the principle of virtual work, may be written as $\iota^*(F) = j^*(\sigma)$.

$$\begin{array}{ccccc}
\boxed{\text{Virtual Velocities}} & & & & \boxed{\text{Local Virtual Velocities}} \\
\| & & & & \| \\
L^{1,\mu}(B, \mathbb{R}^3) & \xleftarrow{\ \iota\ } & L^1_1(\operatorname{Int} B, \mathbb{R}^3) & \xrightarrow{\ j\ } & L^1(\operatorname{Int} B, \mathbb{R}^{12}) \\
 & & & & \\
L^{1,\mu}(B, \mathbb{R}^3)^* & \xrightarrow{\ \iota^*\ } & L^1_1(\operatorname{Int} B, \mathbb{R}^3)^* & \xleftarrow{\ j^*\ } & L^1(\operatorname{Int} B, \mathbb{R}^{12})^* \\
\| & & & & \| \\
L^{\infty,\mu}(B, \mathbb{R}^3) & & & & L^\infty(\operatorname{Int} B, \mathbb{R}^{12}) \\
\| & & & & \| \\
\boxed{\text{Forces}} & & & & \boxed{\text{Stresses}}
\end{array}$$

References

[1] Adams R.A.: Sobolev Spaces, Academic Press, New-York, 1975.
[2] Peterson R.E.: Stress Concentration Factors, Wiley, New-York, 1974.
[3] Segev R.: Generelized Stress Concentration Factors, submitted for publication.