LOW-DIMENSIONAL CHAOTIC DYNAMICS IN DRIPPING FAUCETS

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Summary Chaotic dynamics of the dripping faucet was investigated both experimentally and theoretically. In our experiment using a high-speed camera, we measured continuous change in the position and velocity of the center of mass of the pendant drop prior to its detachment. Continuous trajectories of a low-dimensional chaotic attractor were reconstructed from these data, which was not previously obtained but predicted in our fluid dynamic simulation. From the numerical analysis, we further obtained an approximate potential function with only two variables, the mass of the pendant drop and the position of the center of mass, which corresponds to a set of solutions of Young-Laplace equation. On the basis of the potential landscape we discuss the mechanism of the chaotic dripping faucet.

The rhythm of a dripping water faucet is not always regular and sometimes exhibits irregular behavior, which sensitively depends on the flow rate. It is nowadays well-known that the irregularity of this system arises from deterministic chaos. Most previous studies of the chaotic dripping faucet have involved measuring the time interval T_n between successive drips, because the dripping time is easily measured using a drop-counter apparatus [1]. The time intervals are then plotted in pairs (T_n, T_{n+1}) for each n to give a return map. Because the return maps typically appear low dimensional, the behavior is often described by a simple dynamical model composed of a variable mass and a spring. In this mass-spring model, a mass point, whose mass increases linearly with time at a given flow rate Q, oscillates with a fixed value of the spring constant k; and a part Δm of the total mass m is removed when the spring extension exceeds a threshold, which describes the detachment of a falling drop. Although the model exhibits chaotic return maps similar to those obtained experimentally, its empirical nature means that it does not provide a unified explanation for the complex behavior of the real dripping faucets. Thus, the direct link between the low-dimensional dynamical system and a presumably infinite-dimensional fluid dynamical system remains elusive.

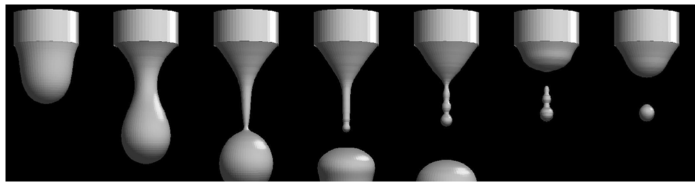

Fig. 1. Simulation results for a dripping water faucet (7 mm diameter).

The aim of this paper is to understand how the dripping faucet dynamics can exhibit low-dimensional chaos. To this end, we tracked the drop formation in continuous time using fluid dynamic simulations based on a new algorithm involving Lagrangian description [2]. Using the simulation, we can reproduce not only the time-dependent shapes of the pendant drops (Fig. 1), but also various characteristics of chaotic dynamical systems [3]. Together with the numerical simulation, we conducted experiments using a high-speed camera. In our experiments, the mass m and the position z_G of the center of mass of the pendant drop under the nozzle was estimated from the shape of the pendant drop using the digitized image recorded by the high-speed video camera every 1/500 s.

Measurement of the continuous-time variables $\{z_G(t), m(t)\}$ made it possible to visualize the reconstructed chaotic attractor in a continuous state space for the first time. The projection of the attractor in the plane (z_G , $\dot{z}_G$) corresponding to a chaotic motion is shown in Fig. 2, including the return map of T_n (Fig. 2 (d)(e)). The top panel of Fig. 2 is the experimentally observed strange attractor, and the bottom panel is numerical simulations. The qualitative agreement between the experiment and the simulation is excellent. On average, $z_G(t)$ oscillated six times during each time interval T_n, which corresponds to the spiral structure of the attractor in Fig. 2. The transition from the oscillating process to the 'necking' process occurs in the region S in which trajectories starting with slightly different remnant masses

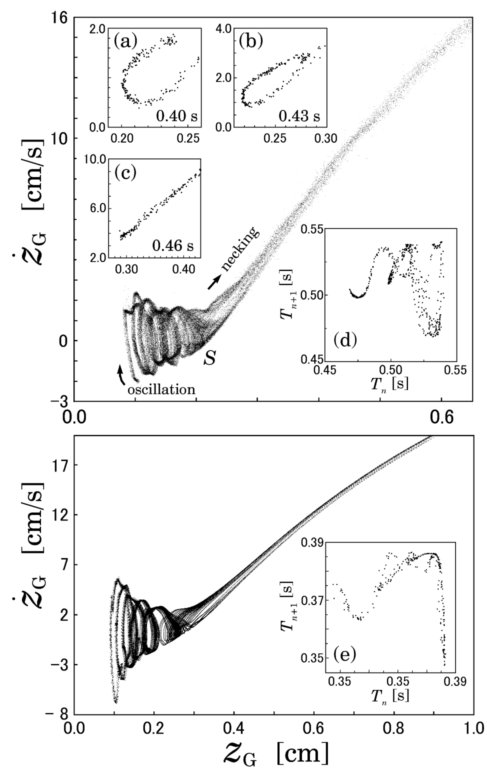

Fig. 2. The top panel is the experimentally observed strange attractor, where the position z_G and the velocity $\dot{z}_G$ of the pendant drop under the nozzle are plotted at every 1/500 s. (a)-(c) Poincaré cross sections of the attractor (plotting of $\dot{z}_G$ vs. z_G at time t measured from each breakup moment. (d) Return map of dripping time intervals T_n. The bottom panel is numerical simulations, where a dripping water faucet (7 mm diameter) is simulated at the flow rate Q = 0.32 g/s. (e) Return map of dripping time intervals T_n.

separate exponentially from each other (Fig. 2 (top)). The cross sections (Fig. 2 (a)-(c)) show the existence of the stretching and folding mechanism, which plays an essential role in chaotic dynamics. The cross sections of the attractor in Fig. 2 (a)-(c) look nearly one-dimensional and explicitly show that the motion of the pendant drop is well characterized by a few state variables. In a wide range of relatively small flow rate (for the nozzle diameter ~ 5 mm, Q < 0.4 g/s), approximately one-dimensional structure of cross sections was observed in both our experiments and numerical simulations.

The motion of the pendant drop is subjected to gravitational force and surface tension. Since the surface energy depends on the shape of the pendant drop, the total potential energy U (i.e., gravitational plus surface) should be a function of the many degrees of freedom of the liquid. Our simulations show, however, that approximating U as a function of two variables, m and z_G, yields a conceptually clear picture for the basic low-dimensional structure of the system. Cross sections of U/m, the potential energy per unit mass, at several fixed values of m for the chaotic attractor shown in Fig. 2 (bottom) are presented in Fig. 3 (a). Although the cross sections in Fig. 3 (a) look two-folded, if they are approximated as single-valued functions, then a sheet of the potential surface $U(m, z_G\})/m$ is obtained as shown in Fig. 3 (b). It can be shown that without solving the fluid dynamical equations the potential function U is obtained from Young-Laplace equation which describes the static equilibrium shape of drops. In 1973, Padday and Pitt proposed a hypothesis that perturbations from one solution of Young-Laplace equation to another are perturbations of the lowest energy [4]. On the basis of this hypothesis, they obtained energy profiles of pendant drops like Fig. 3 (c) from the Young-Laplace equation and discussed stability of the static equilibrium shape. We will demonstrate that the above mentioned potential function U is equivalent to that computed using the Young-Laplace equation. The potential surface is characterized by a U-shaped valley and a ridge which converge as m increases and have totally merged when $m = m_{crit}$, the maximum mass of the static stable shape.

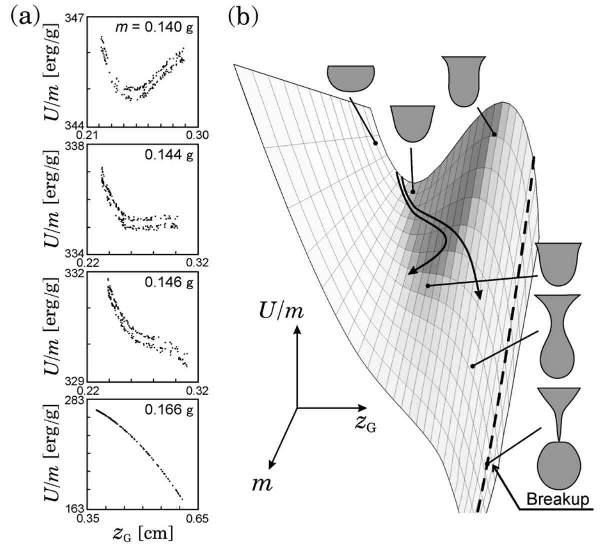

Fig. 3. (a) Fluid dynamic simulation (7 mm diameter, the flow rate $Q = 0.32$ g/s). Cross section of U/m for the attractor is plotted vs. z_G on a constant mass surface, where U/m is the total potential (= surface tension plus gravitation) energy per unit mass as a function of m (pendant drop mass in units of gram in each panel) and z_G (position of the center of mass). (c) Schematic view of a typical potential surface. The pendant drop shapes in various state points (z_G, m) on the potential surface are indicated. The breakup occurs on the dashed line.

Our result leads to the motion of the dripping faucet being described in terms of the approximate potential function U. If the flow rate Q is small enough so that the initial oscillation after the breakup is damped, the state point of the pendant drop goes along the bottom of the valley as m increases. At $m = m_{crit}$, where the valley merges into the ridge, the pendant drop loses its stability (start of the necking process) and rapidly approaches the breakup point (on the broken line in Fig. 3 (b)). Since the instability always begins at the same point $m = m_{crit}$ in this case, the size of each falling drop is uniform (i.e., Tate's law). If, on the other hand, the oscillation of the pendant drop affects the start of necking, the dripping motion exhibits a variety of flow rate-dependent periodic and chaotic patterns. Note that the state point of a pendant drop generally gets over the ridge before m reaches m_{crit} when it is oscillating. At a flow rate resulting in a chaotic motion, two trajectories starting with slightly different m values (different remnant masses) are initially close to each other. As m increases, however, one trajectory may get over the ridge at a certain m value, while the other remains in the valley owing to a small difference in z_G (two lines with arrow in Fig. 3 (b)). The potential landscape illustrates how the trajectories experience stretching and folding as shown in Fig. 2 (a)-(c).

In conclusion, we showed experimentally and numerically the low-dimensionality of the reconstructed chaotic attractor in a continuous state space. Moreover, we demonstrated that the dripping faucet dynamics can be basically described using a potential function with only two variables, the mass of the pendant drop and the position of the center of mass. The potential landscape is the key to understanding how the dripping faucet dynamics can exhibit low-dimensional chaos.

References

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