Three dimensional gravity water waves

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Abstract

Three dimensional gravity water waves are ubiquitous, and of importance to physical oceanography and marine engineering. Yet being a complex nonlinear process the study of water waves remains a rich source of mathematical problems, whose answers can be of relevance to ocean scientists. My presentation concerns the form of three dimensional traveling water waves, contrasting the geometry of waves over deep water with the shallow water regime. In particular, solutions typically occur in two-parameter bifurcation families, which can have complex secondary bifurcations in resonant situations. The rigorous mathematical theory will be compared with numerical computations and controlled laboratory experiments. I will also discuss the existing stability theory, especially of the deep water case.

This paper is concerned with stationary patterns in free surface water waves, which is a classical topic in the fluid dynamics of the ocean surface. Descriptions of two dimensional progressive wavetrains date to at least the time of Stokes, and a rigorous mathematical existence theory was initiated by Levi-Civita [7]. The relevence of these wave forms to observation depends upon their stability to three dimensional perturbations, a topic which is taken up in McLean's two fundamental papers [8][9]. A two dimensional solution of the water wave problem gives rise of course to a three dimensional pattern, namely one which is constant in a second horizontal independent variable. More interesting cases are given by genuinely three dimensional patterns, which is a topic of current research interest, addressed in part in the papers [2][3] of Craig & Nicholls. Solutions arise in two dimensional parameter families, in a bifurcation problem for surfaces whose two governing parameters are the components of the horizontal phase velocity of the solution. An interesting variety of wave patterns emerges, with a distinct change in character between deep water progressive watetrains and wave patterns in shallow water [10]. Again the relevance of these solutions to observations depends upon their stability. Recent experimental observations of such three dimensional wave patterns in deep water are described by Hammack, Henderson & Segur [5]. The goal of the present paper is to describe analytical and numerical results on the stability, or indeed the metastability of the progressive wave patterns given in [2][3][10], as a functions of the principal parameters, namely the mean depth, the aspect ratio of the fundamental horizontal period, steepness, and perhaps most importantly, the particular bifurcation component of the solution. Related stability results in those of Ioualalen, Roberts and Kharif [6].

Our analysis is based on the description of the water waves problem as a Hamiltonian system given by Zakharov [12]. Using the Dirichlet-Neumann operator to express the equations of motion, a convenient form of the Hamiltonian is given in Craig & Sulem [4],

$$H(\eta, \xi) = \int_{\mathbf{T}^2} \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2 \, dx \ , \tag{1}$$

where g is the acceleration due to gravity, $y = \eta(x,t)$ gives the free surface, $\xi(x,t) = \varphi(x, \eta(x,t), t)$ represents the boundary values of the velocity potential on the free surface, and the Dirichlet-Neumann operator [1] is expressed by $G(\eta)\xi dx = N \cdot \nabla\varphi dS$. The torus $\mathbf{T}^2 = \mathbf{R}^2/\Gamma$ is the periodic

*Research partially supported by the Canada Research Chairs Program and the NSERC through grant number 238452-01

fundamental domain of the wave pattern in question. The components of the horizontal momenta for $j = 1, 2$ are given by

$$I_j(\eta, \xi) = \int_{\mathbf{T}^2} \eta \partial_{x_j} \xi \, dx \ . \tag{2}$$

Because the evolution equations take Hamilton's canonical form $\partial_t \eta = \delta_\xi H$, $\partial_t \xi = -\delta_\eta H$, progressive wave patterns follow the classical Lagrange multiplier rule that

$$\delta H = c \cdot \delta I \tag{3}$$

which is to say that the traveling wave patterns are critical points of the Hamiltonian functional (1) for fixed horizontal momenta (2). This forms the basis for a theory of multiplicity of solution branches of progressive waves [3], and in the presence of surface tension a rigorous existence theory [2].

Questions of linear stability involve the linearization of (3) about a traveling wave solution, in a frame of reference moving with the wavetrain. Writing $(\delta\eta, \delta\xi)^T = \exp(\sigma t) v(x)$, $v(x) = (w, z)^T(x)$, the linearization is written

$$\sigma \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \delta^2_{\eta\xi} H & \delta^2_{\eta\eta} H \\ -\delta^2_{\eta\eta} & -\delta^2_{\eta\xi} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \ . \tag{4}$$

This is an eigenvalue problem for pairs v, σ analogous to the Floquet-Bloch band theory of quantum electrons in a crystal lattice. Setting boundary conditions on the fundamental domain $\mathbf{T}^2$ to be $v(x+\gamma) = \exp(ip\cdot\gamma)v(x)$ for all $\gamma \in \Gamma$, we obtain band functions $(v(x;p), \sigma(p)), p \in (\mathbf{T}^2)' = \mathbf{R}^2/\Gamma'$ as solutions to the eigenvalue problem (4). Here Γ' is the dual lattice to $\Gamma \subseteq \mathbf{R}^2$ which determines the periodic fundamental domain. Band functions $\sigma(p)$ with nontrivial real components signify instabilities of the progressive wave pattern at wavenumber p. Given a numerically computed progressive wave pattern, the band functions $\sigma(p)$ and their associates eigenfunctions $v(x;p)$ are computed numerically for indications of such three dimensional instabilities.

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