SHARPENING AND BREAKING OF SUBHARMONIC GRAVITY WAVES ON DEEP WATER

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The purpose of the present work is to investigate the process of sharpening and formation of singular flows in symmetric and non-symmetric subharmonic gravity waves that have a period *m* times greater than the period of Stokes waves. Subharmonic waves are excited due to subharmonic instabilities of Stokes waves and have the crests of different height, as was at first established by Chen and Saffman [1] for symmetric waves and by Zufiria [3] for non-symmetric waves.

Potential two-dimensional gravity waves and related fluid flows in the case of ideal fluid are governed by the following set of equations

Laplace equation:
$$\Delta\Phi = 0\,; \quad -\infty < y < \eta(\theta)\;; \tag{1}$$

Bernoulli equation:
$$-c\Phi_\theta + \frac{1}{2}(\Phi_\theta^2 + \Phi_y^2) + \eta = 0\,, \quad y = \eta(\theta)\,; \tag{2}$$

Kinematic condition:
$$-c\eta_\theta - \Phi_y + \eta_\theta\Phi_\theta = 0\,, \quad y = \eta(\theta)\,; \tag{3}$$

$$\Phi_y = 0\,, \quad y = -\infty\,; \tag{4}$$

where $\Phi = \Phi(\theta, y)$ is the velocity potential, $\eta(\theta)$ is the elevation of the free surface, *c* is the phase speed, $\theta = x - ct$ is the wave phase.

The velocity potential and the elevation of the free surface are looked for in the form of the following truncated Fourier series:

$$R(\theta, y) = \sum_{n=1}^{mN} \xi_{\frac{n}{m}} \exp\left(\frac{n}{m}(y + i\theta)\right), \quad \Phi = -ic(R - R^*)\,; \tag{5}$$

$$\eta(\theta) = \sum_{n=-mM}^{mM} \eta_{\frac{n}{m}} e^{i\frac{n}{m}\theta}, \quad \eta_{-\frac{n}{m}} = \eta_{\frac{n}{m}}^*\,; \tag{6}$$

the coefficients $\eta_{n/m}$ and $\xi_{n/m}$ being real for symmetric waves and complex for non-symmetric waves. The equations of motion are thereby reduced to a system of non-linear algebraic equations for the coefficients of the expansions and the wave phase speed, which is solved by Newton's iterations.

The following results were obtained.

1. The calculations of Chen and Saffman [1] and Simmen and Saffman [2], where integrodifferential equation in the inverse plane formulation was solved, were reproduced using Fourier expansions in the plane of physical variables, the bifurcation points from the Stokes branch having been calculated up to *m* = 10. Then the subharmonic branches for *m* = 2, 3, 5 were traced up to limiting subharmonic waves.

2. Irregular flows beyond limiting subharmonic waves were investigated. These are the flows with stagnation point inside the flow domain and discontinuous streamlines near the wave crests. The horizontal water velocities in such wave crests exceed the speed of the crests themselves, as was established by the authors in the case of Stokes waves [4]. Irregular subharmonic flows were found to originate in the following way. Streamlines become discontinuous only in the highest sharp crests while the flow in all other crests (that are still rounded and have much lesser amplitude) remains regular (see Figure 1 for the case *m* = 2). Irregular flows produce a simple model for describing the initial stage of the formation of spilling breakers when a localized jet is formed at the crest following by generating whitecaps.

The following two alternatives for the exact solutions associated with approximate irregular flows were singled out [5]: (i) the profile of an exact solution is continuous with sharp corner at the crest (a discontinuous first derivative) similar to the limiting Stokes wave but is of lesser steepness; (ii) the profile of an exact solution has two symmetric finite discontinuities in the vicinity of the wave crest and is not single-valued. An additional set of stagnation points is found to exist above the crest area of almost limiting waves supporting the conjecture of Grant [6] that the 120° singularity of the limiting wave is formed of several coalescing 90° singularities.
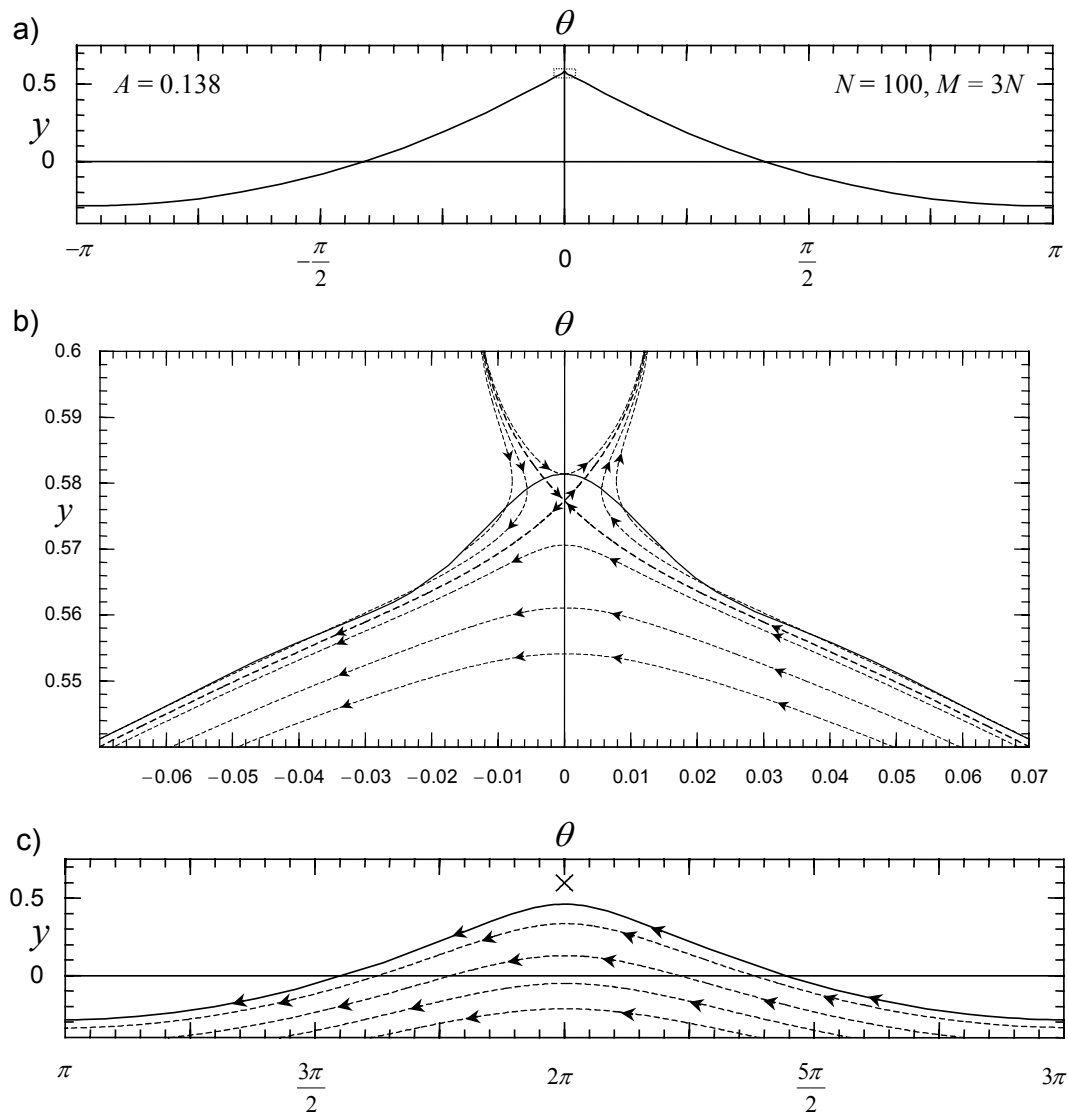
Figure 1. The subharmonic irregular flow of order $m = 2$: a) an *irregular* sharp crest;
b) the irregular flow in the vicinity of the irregular crest; c) a *regular* rounded crest.

3. The effect of non-symmetry of subharmonic waves was found to be more essential with wave sharpening. Non-symmetry can lead directly to wave breaking and formation of subharmonic irregular flows. This produces a good model for describing storm waves with crests of variable height (a ninth roll). In such waves, large-amplitude breaking crests are usually preceded by a number of rounded gentle crests of lesser amplitude.

References

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