UNIQUENESS RESULTS FOR THE REFLECTION-TRANSMISSION PROBLEM

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Summary Reflection and transmission of mechanical waves are investigated for a viscoelastic layer sandwiched between homogeneous elastic half-spaces. On the basis of appropriate boundary conditions for the layer, uniqueness is established for C^2 solutions to the initial/boundary-value problem in the space-time domain.

INTRODUCTION

This paper investigates the reflection and transmission of waves, in the time domain, generated by a viscoelastic (anisotropic) layer sandwiched between homogeneous elastic half-spaces. The problem is regarded as a initial/boundary-value problem for the layer. At least on a interface, both the incident and the reflected/transmitted waves occur simultaneously and hence we cannot pick part of the boundary where the solution is known. This explains why ordinarily existence and/or uniqueness results are lacking in reflection-transmission problems.

The approach presented in this paper follows an energy method and is based on two main steps. First, the boundary conditions for the layer are written in a form which accounts directly for the outgoing character of the (unknown) reflected and transmitted waves. Second, an energy functional is considered for the viscoelastic layer which is a potential for the traction. As a result, uniqueness is established for C^2 solutions in the space-time domain.

Notation and assumptions

Consider a layer of thickness L sandwiched between two half spaces. Let z be the Cartesian coordinate such that $z \in (0, L)$ is the layer and $z < 0$ and $z > L$ are the half spaces. Let $\mathbf{u}(\mathbf{x}, t)$ on $\mathbb{R}^3 \times \mathbb{R}$ be the displacement. We disregard body forces and write the equation of motion as

$$\rho \partial_t^2 \mathbf{u} = \nabla \cdot \mathbf{T}$$

where ρ is the mass density, $\mathbf{T}$ is the symmetric Cauchy stress tensor and ∂_t denotes (partial) time differentiation. To account for viscoelasticity we let $\mathbf{T}$ be given by the gradient of displacement, $\nabla \mathbf{u}$, in the form

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{G}_0(\mathbf{x}) \nabla \mathbf{u}(\mathbf{x}, t) + \int_0^\infty \mathbf{G}'(\mathbf{x}, \eta) \nabla \mathbf{u}(\mathbf{x}, t - \eta) d\eta$$

where the values of $\mathbf{G}_0$ and $\mathbf{G}'$ are fourth-order tensors and $\mathbf{G}'(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^+)$. We also assume that $\nabla \mathbf{u}(\mathbf{x}, \cdot)$, $\partial_t \mathbf{u}(\mathbf{x}, \cdot)$, $\partial_t^2 \mathbf{u}(\mathbf{x}, \cdot) \in L^1(\mathbb{R})$. Both $\mathbf{G}_0$ and $\mathbf{G}'$ are required to satisfy the minor and major symmetries. The traction, at the planes $z =$ constant, is denoted by $\boldsymbol{\tau} = \mathbf{T}\mathbf{e}_3$, $\mathbf{e}_3$ being the unit vector of z. In the elastic half-spaces $z < 0$ and $z > L$ it is $\mathbf{G}' = 0$. In the layer $z \in (0, L)$ both $\mathbf{G}_0$ and $\mathbf{G}'$ are allowed to depend on $\mathbf{x}$ only through z (axial inhomogeneity). Thermodynamics requires that $\mathbf{G}_0$ be positive definite. We also assume that $\mathbf{G}'$ is negative semidefinite.

For isotropic solids the governing equations decouple [1]. The present approach for anisotropic solids leads again to decoupled equations, in the elastic homogeneous half-spaces, by using the eigenvectors of the acoustic tensor.

EQUATIONS FOR ANISOTROPIC SOLIDS AND NORMAL INCIDENCE

Assume that ρ and $\mathbf{G}_0, \mathbf{G}'$ depend on $\mathbf{x}$ through z and $\mathbf{u} = \mathbf{u}(z, t)$, which is the case if the incident wave is normal. Hence the equation of motion becomes

$$\rho \partial_t^2 \mathbf{u} = \partial_z [(\mathbf{Q}_0 + \mathbf{Q}' *) \partial_z \mathbf{u}], \qquad z \in (-\infty, 0) \cup (0, L) \cup (L, \infty),$$

where $*$ means convolution in $\mathbb{R}^+$ and $\mathbf{Q}_0 = \mathbf{e}_3 \mathbf{G}_0 \mathbf{e}_3$ and likewise for $\mathbf{Q}'$; in suffix notation $Q'_{ik} = G'_{i33k}$. As the half-spaces are elastic and homogeneous we can write $\mathbf{Q}' = 0$ and $\mathbf{Q}_0 =$ constant as $z \in (-\infty, 0) \cup (L, \infty)$. Letting $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be the eigenvectors of $\mathbf{Q}_0$ we find that

$$\mathbf{u}(z, t) = \sum_{r=1}^{3} [u_r^f(z, t) + u_r^b(z, t)] \mathbf{a}_r, \qquad z \in (-\infty, 0) \cup (L, \infty)$$

where the superscripts f and b are reminders for forward- and backward-propagating d'Alembert's solutions [2]. Now, for any function $h(z \pm ct)$ we have $\partial_z h \, \partial_t h = \pm c (\partial_z h)^2$. Hence for any component u_r^f and u_r^b we have

$$\partial_z u_r^f \, \partial_t u_r^f \leq 0, \qquad \partial_z u_r^b \, \partial_t u_r^b \geq 0.$$

For definiteness we let the incident wave come from $z<0$ and then the reflected and transmitted waves take the general form

$$\mathbf{u}^R(z,t)=\sum_{r=1}^{3}u_r^b(z,t)\mathbf{a}_r,\qquad z<0,\qquad \mathbf{u}^T(z,t)=\sum_{r=1}^{3}u_r^f(z,t)\mathbf{a}_r,\qquad z>L.$$

Hence, because $\boldsymbol{\tau}=\mathbf{Q}_0\partial_z\mathbf{u}$ we find that the power $\boldsymbol{\tau}\cdot\partial_t\mathbf{u}$ for the reflected and the transmitted waves satisfies the inequalities

$$\boldsymbol{\tau}^R\cdot\partial_t\mathbf{u}^R=\sum_{r=1}^{3}q_r\partial_z u_r^b\,\partial_t u_r^b\geq 0,\qquad \boldsymbol{\tau}^T\cdot\partial_t\mathbf{u}^T=\sum_{r=1}^{3}q_r\partial_z u_r^f\,\partial_t u_r^f\leq 0$$

where q_1,q_2,q_3 are the positive eigenvalues of $\mathbf{Q}_0$, in the pertinent half-space.

Energy functional

Let $\mathbf{u}\in C^2((-\infty,0)\cup(0,L)\cup(L,\infty)\times\mathbb{R}^+)$. For any point $\mathbf{x}$ of the layer, $z\in(0,L)$, and time t, consider the functional

$$\Psi(\partial_z\mathbf{u}(t),\partial_z\mathbf{u}^t)=\frac{1}{2}\partial_z\mathbf{u}(t)\cdot\mathbf{Q}_\infty\partial_z\mathbf{u}(t)-\frac{1}{2}\int_0^\infty[\partial_z\mathbf{u}(t-\xi)-\partial_z\mathbf{u}(t)]\cdot\mathbf{Q}'[\partial_z\mathbf{u}(t-\xi)-\partial_z\mathbf{u}(t)]d\xi$$

the dependence on $\mathbf{x}$ being understood and not written. A direct calculation shows that Ψ is the potential for the traction,

$$\frac{\partial\Psi}{\partial[\partial_z\mathbf{u}^t]}=\mathbf{Q}_0\partial_z\mathbf{u}(t)+\int_0^\infty\mathbf{Q}'(\xi)\partial_z\mathbf{u}(t-\xi)d\xi=\boldsymbol{\tau}(t).$$

Consider the energy $E(t)$ for the layer in the form

$$E(t)=\int_0^L\Big\{\frac{1}{2}\rho[\partial_t\mathbf{u}(t)]^2+\Psi(\partial_z\mathbf{u}(t),\partial_z\mathbf{u}^t)\Big\}dz.$$

The assumption that $(\mathbf{Q}')'$ be positive semidefinite, integrations by parts and use of the symmetry of $\mathbf{Q}_0,\mathbf{Q}'$ allow us to find that the time derivative $\dot{E}$ is bounded by

$$\dot{E}(t)\leq\boldsymbol{\tau}(L_-,t)\cdot\partial_t\mathbf{u}(L_-,t)-\boldsymbol{\tau}(0_+,t)\cdot\partial_t\mathbf{u}(0_+,t).$$

UNIQUENESS FOR THE REFLECTION-TRANSMISSION PROBLEM

The reflection-transmission problem P consists in finding a function $\mathbf{u}(z,t)\in C^2((-\infty,0)\cup(0,L)\cup(L,\infty)\times\mathbb{R}^+)$ such that $\mathbf{u}(z,t)=0, z\in\mathbb{R}, t\leq 0$ whereas $\mathbf{u},\boldsymbol{\tau}$ are continuous everywhere and the incident wave $\mathbf{u}^I$ is known so that $\mathbf{u}^I(0_-,t)=\mathbf{w}(t)$, $t\geq 0$. Uniqueness is proved first for the layer $z\in(0,L)$ and next for the half-spaces $z<0$, $z>L$.

Theorem 1 (Layer) *The restriction to $z\in(0,L)$ of the solution $\mathbf{u}$ to P is unique in $C^2((0,L)\times\mathbb{R}^+)$.*
Proof. Let $\mathbf{u}_1,\mathbf{u}_2$ be two solutions to P and $\boldsymbol{\tau}_1,\boldsymbol{\tau}_2$ the associated tractions. The differences $\mathbf{v}=\mathbf{u}_1-\mathbf{u}_2$ and $\boldsymbol{\sigma}=\boldsymbol{\tau}_1-\boldsymbol{\tau}_2$ satisfy the equation $\partial_z\boldsymbol{\sigma}=\rho\partial_t^2\mathbf{v}$ as $z\in(0,L)$, $t>0$, the initial condition $\mathbf{v}(z,0)=0$, $z\in[0,L]$ and the inequalities $\boldsymbol{\sigma}\cdot\partial_t\mathbf{v}\geq 0$ as $z=0_+$ and $\boldsymbol{\sigma}\cdot\partial_t\mathbf{v}\leq 0$ as $z=L_-$. The energy of the layer $E(t)$ associated with $\mathbf{v}$ is shown to satisfy $E(t)\geq 0$, $E(0)=0$, and $\dot{E}\leq 0$. This implies the vanishing of E in $\mathbb{R}^+$ and hence the vanishing of $\mathbf{v}$ in $[0,L]\times\mathbb{R}^+$. This in turn implies uniqueness. □
Because $\mathbf{u}$ is unique as $z\in[0,L]$ and $\mathbf{u}$ is continuous then $\mathbf{u}(0_-,t)$ and $\mathbf{u}(L_+,t)$ are unique.

Theorem 2 (Half-spaces) *For every finite $T>0$ the solution $\mathbf{u}$ to P subject to $\mathbf{u}(0_-,t)=\psi(t)$, $\mathbf{u}(L_+,t)=\boldsymbol{\zeta}(t)$, $t\leq 0$ and $\mathbf{u}(z,0)=\mathbf{l}(z)$, $z\in(-\infty,0]\cup[L,\infty)$, whereas $\mathbf{u}(z,t)$ has compact support as $t\leq T$, is unique in $C^2((-\infty,0)\cup(L,\infty)\times[0,T])$.*
Proof. The difference $\mathbf{v}$ of two solutions has compact support and is subject to $\mathbf{v}(0_-,t)=0$, $\mathbf{v}(L_+,t)=0$, $t\geq 0$, $\mathbf{v}(z,0)=0$, $z\in(-\infty,0_-)\cup(L_+,\infty)$. Consider the energy of the half-space $z<0$, $E_-(t)$, associated with $\mathbf{v}$. We find that $\dot{E}_-(t)=\boldsymbol{\sigma}(0_-,t)\cdot\partial_t\mathbf{v}(0_-,t)=0$. Because $E_-(t)\geq 0$ and $E_-(0)=0$ it follows that $E_-(t)=0$, $t\in[0,T]$. Hence we conclude that $\mathbf{u}$ is unique as $z\in(-\infty,0]$. Likewise we establish uniqueness for $z\in[L,\infty)$. □

Uniqueness is shown to hold also if the half-space $z>L$ is replaced by a fixed boundary, $\mathbf{u}(L,t)=0$, or a free boundary, $\boldsymbol{\tau}(L,t)=0$.

References

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[2] Caviglia G., Morro A.: Reflection and transmission of transient waves in anisotropic elastic multilayers, Quart. Jl Mech. Appl. Math., in print.