CURVATURE INSTABILITY OF A VORTEX RING

Yasuhide Fukumoto*, Yuji Hattori**
*Graduate School of Mathematics, Kyushu University, Japan
**Faculty of Engineering, Kyushu Institute of Technology, Japan

Summary A new instability mechanism is found for Kelvin's vortex ring, which may surpass the Widnall instability. The effect of ring curvature emerges at $O(\epsilon)$ in the asymptotic solution of the Euler equations in powers ϵ, the ratio of core to ring radii. We show that the $O(\epsilon)$ field causes a parametric resonance between a pair of Kelvin waves whose azimuthal wavenumbers are separated by 1. A closed-form solution enables us to calculate the maximum growth rate to be $165/256\epsilon$ and to make headway to nonlinear stability.

MOTIVATION AND MAIN RESULT

Vortex rings are invariably susceptible to wavy distortions, leading to violent wiggles and sometimes to disruption. It has been widely accepted that the *Moore-Saffman-Tsai-Widnall instability* (the *MSTW instability*) is responsible for genesis of unstable waves [1]–[5]. Remember that this is an instability for a straight vortex tube subjected to a straining field.

When viewed locally, a thin vortex ring looks like a straight tube. Because of circular-cylindrical symmetry, the *Rankine vortex*, a circular core of uniform vorticity, is neutrally stable and supports a family of three-dimensional oscillations called the *Kelvin waves*. The vortex ring induces, on itself, not only a local uniform flow that drives itself but also a local straining field akin to a pure shear [1]. This is a quadrupole field proportional to $\cos 2\theta$ and $\sin 2\theta$, in terms of local polar coordinates (r, θ) in the meridional plane, with its origin at the core center and with $\theta = 0$ along the traveling direction . This field breaks the circular symmetry of the core by deforming it into ellipse, and feeds parametric resonance between two Kelvin waves whose azimuthal wavenumbers are separated by 2.

However this might be a oversimplified picture. The asymptotic solution of the Navier-Stokes or the Euler equations for a thin vortex ring in powers of a small parameter ϵ, the ratio of core- to ring-radii, starts with a circular-cylindrical tube at $O(\epsilon^0)$. A vortex ring is featured by vortex-lines curvature This feature manifests itself, at $O(\epsilon^1)$, as a local *dipole field* proportional to $\cos\theta$ and $\sin\theta$. The quadrupole field comes merely as a correction at $O(\epsilon^2)$ [3, 6, 7]. The dipole field also acts as a symmetry-breaking perturbation. Despite its dominance, this has not attracted as much attention as it deserves.

In the present investigation, we explore a possible instability that the dipole field can trigger. We show that the dipole field causes a parametric resonance between two Kelvin waves whose azimuthal wavenumbers differ by 1.

Remarkably we have succeeded in constructing *an explicit solution* of the linearized Euler equations, in terms of the Bessel and the modified Bessel functions. Thereby, an accurate computation of the growth rate becomes feasible for all azimuthal wavenumber combinations $(m, m+1)$ of Kelvin waves. The closed-form solution is amenable to an asymptotic analysis. We reveal that that the most unstable mode occurs in the short-wave limit with radial are azimuthal wavenumbers being of the same magnitude, with the maximum growth rate $165/256\,\epsilon$. The same value has been reached by the geometric optics method [8]. Contrary to the MSTW instability, all of multiple eigenvalues do not result in resonance. This discrepancy is accounted for by Krein's theory of Hamiltonian spectra [9], with the aid of the formula for energy of Kelvin waves [5].

SETTING OF LINEAR STABILITY PROBLEM

A formulation of the linear stability analysis was performed by Widnall & Tsai [3], but the dipole effect has gone untouched.
The center circle penetrating inside the toroidal ring is parameterized by the arclength s. Kelvin's vortex ring is an axisymmetric solution of the Euler equations valid to $O(\epsilon)$. The r and θ components of velocity field inside the core are written, after an appropriate nondimensionalization, as

$$U = \epsilon U_1(r,\theta) + \cdots\,, \quad V = V_0(r) + \epsilon V_1(r,\theta) + \cdots\,; \quad V_0 = r\,, \;\; U_1 = \frac{5}{8}(1-r^2)\cos\theta\,, \;\; V_1 = \left(-\frac{5}{8} + \frac{7}{8}r^2\right)\sin\theta\,.$$

Upon this, we superimpose the following form of the disturbance velocity

$$\tilde{\boldsymbol{v}} = (\boldsymbol{v}_0 + \epsilon \boldsymbol{v}_1 + \cdots)e^{i(ks-\omega t)}\,; \qquad k = k_0 + \epsilon k_1 + \cdots\,, \quad \omega = \omega_0 + \epsilon\omega_1 + \cdots\,.$$

Suppose that a pair of Kelvin waves whose azimuthal wavenumbers differ by 1 are simultaneously excited to $O(\epsilon^0)$:

$$\boldsymbol{v}_0 = \boldsymbol{v}_0^{(1)} e^{im\theta} + \boldsymbol{v}_0^{(2)} e^{i(m+1)\theta}\,.$$

Then the wave excited at $O(\epsilon)$ is found from the linearized Euler equations to possess the following angular dependence

$$\boldsymbol{v}_1 = \boldsymbol{v}_1^{(1)} e^{im\theta} + \boldsymbol{v}_1^{(2)} e^{i(m+1)\theta} + \boldsymbol{v}_1^{(3)} e^{i(m-1)\theta} + \boldsymbol{v}_1^{(4)} e^{i(m+2)\theta}\,.$$

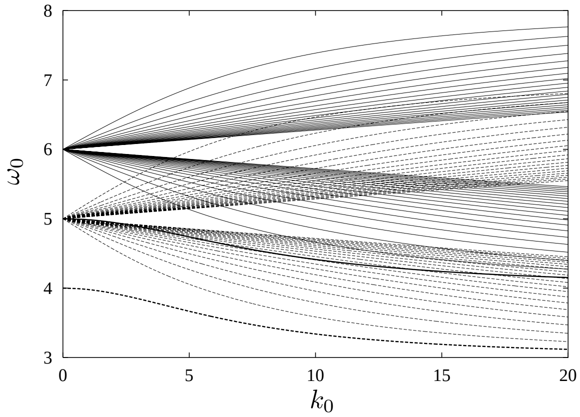

Figure 1. Dispersion relation of Kelvin waves of $m = 5$ (dashed lines) and $m = 6$ (solid lines).

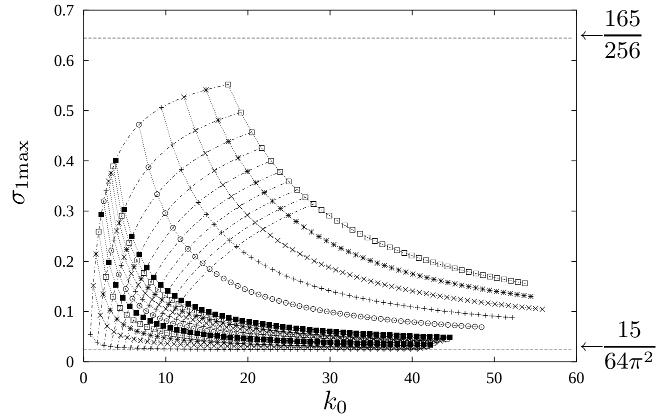

Figure 2. Growth rate of the principal modes.

Excitation, at $O(\epsilon)$, of a pair of waves with the same azimuthal wavenumbers as at $O(\epsilon^0)$ implies a possibility of parametric resonance. In practice, imposition of the boundary conditions at the edge of the core ($r = 1$) yields solvability conditions including the amplitude of the m and the $m + 1$ waves of $O(\epsilon^0)$. Simultaneous excitation of both waves are requisite for instability ($\mathrm{Im}\,\omega_1 > 0$), and a combination of the conditions for the m and the $m + 1$ waves give rise to the growth rate $\sigma_1 = |\mathrm{Im}\,\omega_1|$.

NUMERICAL RESULTS AND SHORT-WAVELENGTH ASYMPTOTICS

Instability is permissible only at the intersection points, in the (k_0, ω_0) plane, of the dispersion curves of the m and the $m+1$ waves. In case of instability, resonance occurs in a small wavenumber band of width of $O(\epsilon)$ around the intersection point $k = k_0$ and the growth rate takes its local maximum value $\sigma_{1\max}$ at $k = k_0$.

Figure 1 illustrates the dispersion relation of Kelvin waves of $m = 5$ (dashed lines) and $m = 6$ (solid lines). Both waves consist of infinitely many branches. The growth rate $\sigma_{1\max}$ is calculated at many of the intersection points. Destabilization occurs only at the intersection points between upgoing modes of $m = 5$ and downgoing modes of $m = 6$. Relatively large growth rate is maintained to short wavelengths at intersection points of branches with the same labels, which we call the *principal mode*.

Calculation of the growth rate for the principal modes is extended to a large azimuthal wavenumber m in figure 2. The same symbol is used for the same azimuthal wavenumber pair $(m, m + 1)$, and the lowest sequence (symbol $+$) corresponds to $m = 0$. Given $(m, m + 1)$, the growth rate decreases with k_0 and tends to $\sigma_{1\max} = 15/64\pi^2 \approx 0.02374715242$ as $k_0 \to \infty$. On the other hand, given the branch label, the growth rate increases monotonically with m and approaches the common value $\sigma_{1\max} = 165/256 = 0.64453125$ as $m \to \infty$. By virtue of the closed-form solution for $\boldsymbol{v}_1^{(1)}$ and $\boldsymbol{v}_1^{(2)}$, the asymptotics for the first principal mode (the leftmost curve) is deduced as

$$\sigma_{1\max} \approx 0.64453125 - 1.548698742/m^{2/3}.$$

This is the most dominant mode over the all possible resonance pairs. This mode outweighs, for the entire range of ϵ ($0 < \epsilon \lesssim 1$), the Widnall instability which is of $O(\epsilon^2)$.

The disturbance vorticity field is calculated and its correlation with the local strain is examined. The instability mechanism is traced to stretching of disturbance vortex lines in the toroidal direction. Discussions are also given to the effects of viscosity and nonlinearity.

References

[1] Widnall, S. E., Bliss, D. B., Tsai, C.-Y.: The instability of short waves on a vortex ring. *J. Fluid Mech.* **66**: 35–47, 1974.
[2] Moore, D. W., Saffman, P. G.: The instability of a straight vortex filament in a strain field. *Proc. R. Soc. Lond. A* **346**: 413–425, 1975.
[3] Widnall, S. E., Tsai, C.-Y.: The instability of the thin vortex ring of constant vorticity. *Phil. Trans. R. Soc. Lond. A* **287**: 273–305, 1977.
[4] Eloy, C. & Le Dizès, S.: Stability of the Rankine vortex in a multipolar strain field. *Phys. Fluids* **13**: 660–676, 2001.
[5] Fukumoto, Y.: The three-dimensional instability of a strained vortex tube revisited. *J. Fluid Mech.* **493**: 287–318, 2003.
[6] Fukumoto, Y., Moffatt, H. K.: Motion and expansion of a viscous vortex ring. Part 1. A higher-order asymptotic formula for the velocity. *J. Fluid Mech.* **417**: 1–45, 2000.
[7] Fukumoto, Y.: Higher-order asymptotic theory for the velocity field induced by an inviscid vortex ring. *Fluid Dyn. Res.* **30**: 67–95, 2002.
[8] Hattori, Y., Fukumoto, Y.: Short-wavelength stability analysis of thin vortex rings. *Phys. Fluids*. **15**: 3151–3163, 2003.
[9] Arnold, V. I.: Mathematical Methods of Classical Mechanics, 2nd ed., Springer-Verlag, 1989.