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VISCOELASTIC FLUID FLOWS IN A FALLING CYLINDER VISCOMETER AND THE EVALUATION OF SHEAR VISCOSITY

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Summary In this paper we consider the problem of the flow of some non-newtonian fluids (second grade, third grade and second order fluids) in a falling cylinder viscometer. If the problem for a second grade fluid is similar to the corresponding problem for linear viscous fluids, largely discussed in Cristescu and all (2002), the other two problems are essentially different. The differences (appearing in the velocity field and shear stress component) are put into evidence. More, the formula of the shear viscosity for a third grade fluid can be interpreted (for instance) as a relation for the determination of the sum $\beta_2 + \beta_3$ of constitutive moduli (and leads, in the case of the law obtained by Fosdick and Rajagopal (1980), to the complete determination of constitutive moduli).

POSITION OF THE PROBLEM

The problem of the falling cylinder viscometer filled with a linear incompressible viscous fluid was recently analyzed in Cristescu and all [1]. Here it was presented the theory of this apparatus, some experiments and obtained an analytical solution, for the flow problem and a formula for the shear viscosity (which, in this case, coincides with the kinematic viscosity). An extensive bibliography is also presented and discussed in this paper and we refer to this one. However many biological fluids have a significant non-newtonian behaviour and for this reason we consider here the problem for some classes of non-newtonian fluids like: second grade, third grade and second order fluids. We rely on the description of the apparatus made in [1]. We emphasize here that the radius of the falling cylinder R_1 is of the order of 10^{-4} m, while the radius of the outer (fixed) cylinder, R_2 is: $R_1 < R_2 < 10^{-3}$ m. The velocity of the falling cylinder is of the order of $10^{-4} - 10^{-3}$ m/s, l « h, where l is the length of the falling (inner) cylinder and h is the length of the outer cylinder. We also remind that, in viscoelastic fluids, the shear viscosity is more significant than in a linear viscous fluid (and generally depends on shear rate). We remark on the one hand that, for second grade fluids, the problem is similar to those described in [1] for a linear viscous fluid and, on the other hand that, for third grade fluids, as well as for second order fluids, the problems are completely different. The existence and uniqueness of the solutions are proved and numerical calculations are performed in order to (graphically) compare the velocity fields and the shear stress components. A new formula (after our knowledge) which connect the shear viscosity, the plateau viscosity and the constitutive moduli β_2, β_3 is obtained, for a third grade fluid (which is useful in order to evaluate the sum $\beta_2 + \beta_3$). We underline (see also [1]) that the measurements must be made into the "central" part (in respect to the length of the outer cylinder) of the flow such as to provide for V_1 a constant value during the experiment (see [1] Fig.2). The admissible velocity field considered here is (in cylindrical coordinates)

$$v_r = 0, \quad v_\theta = 0, \quad v_z = V_1 f(r), \ r \in [R_1, R_2]. \tag{1}$$

and the boundary conditions are

$$v_z(R_1) = V_1, \quad v_z(R_2) = 0. \tag{2}$$

From this point we suppose that (due to the fact that h » l, the discussions from [1] and the introduction to [2]) the pipe (outer fixed cylinder) has of infinite length and we choose the reference (cylindrical) system with the z-axis oriented in the same sense as the acceleration of gravity. As the considered motion is viscometric Cauchy's stress tensor will be given by $\mathbf{T} \equiv -p\mathbf{I} + \overline{\mathbf{T}}(\mathbf{A_1}, \mathbf{A_2})$ and we consider the following formulae for second grade (see Dunn and Fosdick [3]), third grade (see Tigoiu [4], with the mention that $\mathbf{A_3} = \mathbf{0}$, or Fosdick and Rajagopal [5] for a different formula) and second order fluids ([3])

$$\begin{aligned} \overline{\mathbf{T}}(\mathbf{A_1}, \mathbf{A_2}) = \;& \mu\mathbf{A_1} + \alpha_1(\mathbf{A_2} - \mathbf{A_1}^2), \quad \mu, \ \alpha_1 \in \mathbf{R}, \ \mu \geq 0; \\ \overline{\mathbf{T}}(\mathbf{A}_1, \mathbf{A}_2) = \;& \mu\mathbf{A}_1 + \alpha_1(\mathbf{A}_2 - \mathbf{A}_1^2) + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \\ & \mu \geq 0, \ \beta_1 < 0, \ \beta_1 + 2(\beta_2 + \beta_3) \geq 0; \\ \overline{\mathbf{T}}(\mathbf{A_1}, \mathbf{A_2}) = \;& \mu(tr\mathbf{A_1^2})\mathbf{A_1} + \alpha_1(\mathbf{A_2} - \mathbf{A_1}^2), \quad \alpha_1 \in \mathbf{R}, \\ & \mu(tr\mathbf{A_1^2}) = \eta_0 exp\left\{ -A ln10 \left[\frac{1}{2} log\left(\frac{R_1^2}{V_1^2} tr\mathbf{A_1^2} \right) + 3 \right]^2 \right\}. \end{aligned} \tag{3}$$

Here, for the formula for a second order fluid, we have employed (for the choice of shear viscosity expression $(3)_4$) the graphs from Coleman, Markowitz and Noll [6], for a polyisobutylene in decaline solution (13%) in order to obtain A = 5/64. η_0 can generally be considered as the plateau viscosity.

SOLUTIONS FOR THE BOUNDARY VALUE PROBLEM AND EVALUATION OF SHEAR VISCOSITY

Introducing the admissible velocity field (1) into (3) and the resulting formulae into the balance of linear momentum equations we obtain (adding also the boundary conditions (2)) the corresponding flow problems. For the second grade fluid, the solution of the boundary value problem is analytically obtained and for the dimensional velocity field and for the shear stress component we have

$$\begin{aligned} &\mu v_z(r) = \frac{1}{4}(\frac{\partial p}{\partial z} - \rho_l g)r^2 + \mu V_1 C_1 ln(r/R_1) + \mu V_1 C_2; \\ &T_{rz}(r) = \frac{1}{2}(\frac{\partial p}{\partial z} - \rho_l g)r + \mu V_1 C_1 \frac{1}{r}; \\ &C_1 = -[\frac{1}{4}Re\sigma_z \frac{R_2^2 - R_1^2}{R_1^2} + 1]\frac{1}{ln(R_2/R_1)}; \quad C_2 = 1 - \frac{1}{4}Re\sigma_z; \\ &Re \equiv \frac{\rho_l g R_1^2}{\mu V_1}, \quad \sigma_z \equiv \frac{\partial p}{\partial z} - \rho_l g, \end{aligned} \tag{4}$$

where ρ_l is the mass density of the fluid. It is simply to remark that in this case the shear viscosity $\eta = (\mu) = const.$ (unfortunately) and the solution is slightly different from the relation (3.15) in [1].
For a third grade fluid $(3)_2$, a first integral for the flow problem leads to the determination of the nondimensional shear stress by

$$f' + \beta f'^3 = (T_{rz}) = \frac{1}{2}Re\sigma_z r + C_1 \frac{1}{r} \tag{5}$$

and the unique solution of equation (5) has the form

$$f'(r) = \frac{1}{(2\beta)^{\frac{1}{3}}}\left\{\sqrt[3]{C + \sqrt{C^2 + \frac{4}{27\beta}}} + \sqrt[3]{C - \sqrt{C^2 + \frac{4}{27\beta}}}\right\}, \tag{6}$$

where we have denoted $C \equiv C(r) \equiv ar + \frac{C_1}{r}$, $a \equiv \frac{1}{2}Re\sigma_z$ and $\beta \equiv \frac{2(\beta_2 + \beta_3)V_1^2}{\mu R_1^2}$. The solution of (6) was obtained by numerical integration and graphically compared with the corresponding solution for a second grade fluid (or a linear viscous fluid). From the formula for the shear viscosity, computed for $r = R_1$, for instance, we obtain the relation

$$\eta = \mu(1 + \beta f'^2(1;\beta)), \tag{7}$$

for the determination of β, once the plateau viscosity, η_0, is known (here $\eta_0 = \mu$).
For a second order fluid $(3)_{3,4}$, the first integral (giving the shear stress component) is

$$\mu(f'^2)f' = (T_{rz}) = \tfrac{1}{2}\mathrm{Re}\sigma_z r + C_1 \tfrac{1}{r}. \tag{8}$$

This equation can be written in first approximation as

$$f'(r) = (\tfrac{1}{2}\mathrm{Re}\sigma_z r + C_1\tfrac{1}{r})\left\{1 + \mathrm{A}ln10\left[\tfrac{1}{2}log f'^2 + 3\right]^2\right\}. \tag{9}$$

On a part this last equation (by combining with a shooting method) can be numerically solved (and the solution is graphically compared with those for a third grade and second grade fluid). On an other part, restricting ourself to the first approximation of (9), we obtain a forth order equation with an unique negative solution (f'<0), which finally is again numerically computed.

References

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