FUNDAMENTAL INEQUALITIES FOR THE BOUNDS ON THE EFFECTIVE TRANSPORT COEFFICIENTS OF TWO–PHASE MEDIA

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Summary By starting from the truncated power expansions given at a number of discrete real points we establish the general inequalities for the effective transport coefficients Q of macroscopically isotropic two phase media.The inequalities obtained provide the new upper and lower bounds on Q the best with respect to the available power series coefficients. For specific cases these new bounds reduce to the classical ones due to Wienner and Hashin-Shtrikman. They are also coincide with the estimations of Bergman [1] and Milton [3] derived for the fitting problem. Many illustrative examples show the usefullness of the results obtained..

FORMULATION OF THE PROBLEM

It is well-known that the effective transport coefficient of a two-phase medium with macroscopic isotropic symmetry has a Stieltjes integral representation:

$$f_1(x) = \frac{Q(x)-1}{x} = \int_0^1 \frac{d\gamma_1(u)}{1+xu},\ d\gamma_1(u) > 0,\ x = \frac{q_2}{q_1} - 1,\ Q(x) = \frac{q_{ef}(x)}{q_1};\ f_1(-1) \leq 1. \tag{1}$$

Here q_{ef} denotes the effective modulus of the composite material, while q_1, q_2 describe the physical properties of its constituents. Assume that the truncated power expansions of $f_1(x)$ at $x_1,\ x_2, ..., x_N, -1$ be given, where $-1 < x_j,\ j = 1, 2, ..., N,$

$$f_1(x) = f_1(-1) + O(x+1);\ f_1(-1) \leq 1;\ f_1(x) = \sum_{i=0}^{p_j-1} c_{ij}(x-x_j)^i + O((x-x_j)^{p_j}),\ j = 1, 2, ..., N\ . \tag{2}$$

The following abbreviated notation for the truncated power series (2) will also be used:

$$f_1(x) = f_1|_{\boldsymbol{x}}^{\boldsymbol{p}}(x-\boldsymbol{x}) + O(x-\boldsymbol{x})^{\boldsymbol{p}},\ \boldsymbol{x} = [x_1, x_2, ..., x_N, -1]\ ,\ \boldsymbol{p} = [p_1, p_2, ..., p_N, 1]\ ,\ f_1(-1) \leq 1 \tag{3}$$

The aim of this contribution is to establish, starting from the power expansions (3), the general inequalities for the Stieltjes function $f_1(x)$, and next via (1) the best upper and lower bounds on the effective coefficient $Q(x)$.

THE METHOD OF SOLUTION

In the first step the inequality $f_1(-1) \leq 1$ is replaced formally by the equality $f_1(-1) = 1$, cf. (3). The input data (3) transform to

$$f_1(x) = f_1|_{\boldsymbol{x}}^{\boldsymbol{p}}(x-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}},\ \boldsymbol{x} = [x_1, x_2, ..., x_N, -1]\ ,\ \boldsymbol{p} = [p_1, p_2, ..., p_N, 1]\ ,\ f_1(-1) = 1. \tag{4}$$

Next by introducing $P_0 = 0,\ P_j = \sum_{i=1}^{j} p_i,\ j = 1, 2, ..., N;\quad P = P_N + 1$ we expand the original Stieltjes function $f_1(x)$ given by (1) to the new Stieltjes one $f_P(x)$

$$\begin{aligned}
&f_{P_0+1}(x) = \frac{f_{P_0+1}(x_1)}{1+(x-x_1)f_{P_0+2}(x)},\ f_{P_0+2}(x) = \frac{f_{P_0+2}(x_1)}{1+(x-x_1)f_{P_0+3}(x)},\ ...,\ f_{P_1}(x) = \frac{f_{P_1}(x_1)}{1+(x-x_1)f_{P_1+1}(x)},\\
&f_{P_1+1}(x) = \frac{f_{P_1+1}(x_2)}{1+(x-x_2)f_{P_1+2}(x)},\ f_{P_1+2}(x) = \frac{f_{P_1+2}(x_2)}{1+(x-x_2)f_{P_1+3}(x)},\ ...,\ f_{P_2}(x) = \frac{f_{P_2}(x_2)}{1+(x-x_2)f_{P_2+1}(x)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&f_{P_{N-1}+1}(x) = \frac{f_{P_{N-1}+1.}(x_N)}{1+(x-x_{N-1})f_{P_{N-1}+2.}(x)}, ..,\ f_{P_N}(x) = \frac{f_{P_N}(x_N)}{1+(x-x_N)f_P(x)},\ f_P(x) = f_P(-1) + O(x+1).
\end{aligned} \tag{5}$$

It is convenient to rewrite (5) as follows

$$f_1(x) = f_1{}_{\boldsymbol{x}}^{\boldsymbol{p}}(x,\ f_P(x)) = f_1{}_{x_1,x_2,...,x_N,1}^{p_1,p_2,...,p_N,1}(x,\ f_P(x)),\ \text{where}\ f_P(x) = \int_0^1 \frac{d\gamma_P(u)}{1+xu}\ \text{and}\ f_P(-1) = w_p. \tag{6}$$

For any values of $f_P(x) \in [0, w_p]$, $x \in [-1, \infty]$, the continued fraction expansion $f_1{}_{\boldsymbol{x}}^{\boldsymbol{p}}(x,\ f_P(x))$ satisfies the input data (4). Let us introduce now the function $F_P(x,u)$ estimating $f_1(x)$ in the following way

$$f_1(x) \in \{F_P(x,u);\ -1 \leq u \leq 0\}\ ,\ \text{where}\ F_P(x,u) = f_1{}_{x_1,x_2,...,x_N,1}^{p_1,p_2,...,p_N,1}(x,\ w_P\,(1+u)), \tag{7}$$

The sign of the first derivative of $F_P(x,u)$ with respect to u, obtained after lengthy calculations is equal to:

$$\text{sign}\left(\frac{\partial F_P(x,u)}{\partial u}\right) = (-1\ )^0,\ = (-1)^{P_1},\ \cdots,\ = (-1)^{P_N}\ \text{if}\ -1 < x < x_1,\ x_1 < x < x_2, \cdots,\ x_N < x < \infty, \tag{8}$$

where $P_j = \sum_{i=1}^{j} p_i,\ j = 1, 2, ..., N;\ \ P = P_N + 1$. From (8) the inequalities follow:

$$\begin{array}{lllll}
(-1)^0\ F_P(x,-1) & \leq\ (-1)^1\ f_1(x)\ \leq & (-1)^1\ F_P(x,0) & \text{if} & -1 < x \leq x_1,\\
(-1)^{P_1} F_P(x,-1) & \leq (-1)^{P_1+1} f_1(x) \leq & (-1)^{P_1+1} F_P(x,0) & \text{if} & x_1 \leq x \leq x_2,\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots\\
(-1)^{P_N} F_P(x,-1) & \leq (-1)^{P_N+1} f_1(x) \leq & (-1)^{P_N+1} F_P(x,0) & \text{if} & x_N \leq x < \infty.
\end{array} \tag{9}$$

The functions $F_P(x,0)$ and $F_P(x,-1)$ are the multipoint Padé approximant $[m_P/n_P](x)$ and $[m_{P-1}/n_{P-1}](x)$ to $f_1(x)$, where $[m_P/n_P] = \frac{a_0+a_1x^1+\dots+a_{m_P}x^{m_P}}{1+b_1x^1+\dots+b_{n_P}x^{n_P}}$, $m_P = P-1-n_P$, $n_P = E(P/2)$, $P = \sum_{j=1}^{N} p_j + 1$. Relations (9) provide the fundamental inequalities for the multipoint Padé approximants to the Stieltjes function $f_1(x)$ representing via (1) the effective transport coefficient $Q(x)$.

FUNDAMENTAL INEQUALITIES

Let $L_R(x) = \sum_{j=1}^{N} p_j H(x-x_j)$ determining the total number $p_1+p_2+...p_s$ of the coefficients of the power expansions of $f_1(x)$ available at points $x_1,\ x_2,\ ...,x_s \le x$ be given, see (4). By introducing the piecewise continnous function $M_R(x) = (L_R(x)$ if $-\infty < x < 0$ or $L_R(x)+1$ if $-\infty < x < 0)$, the relations (9) can be easily transformed to the following inequalities for the multipoint Padé approximants $x[m_R/n_R]$ and $x[m_{R+1}/n_{R+1}]$ to the effective transport coefficient $Q(x)-1$, cf. (1),

$$(-1)^{M_R(x)}x[m_{R+1}/n_{R+1}](x) \le (-1)^{M_R(x)}\left(Q(x)-1\right) \le (-1)^{M_R(x)}x[m_R/n_R](x),\ x\in[-1,\infty). \quad (10)$$

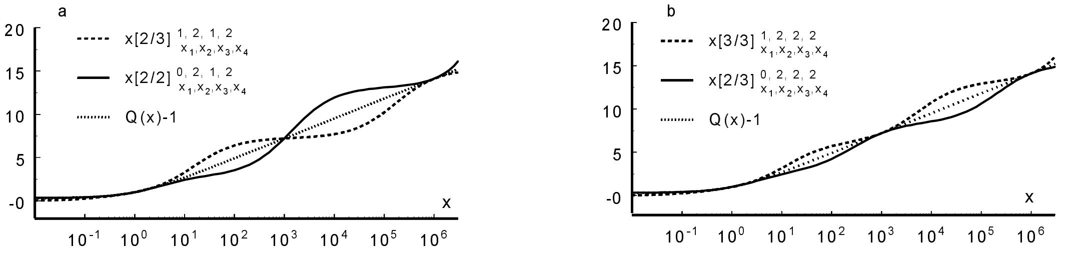

Fig. 1: Three- and four-point Padé approximants to the function $Q(x)-1 = xf_1(x)$, $f_1(x) = \frac{\ln(0.5(2+x))}{x}$. representing the bounds on $Q(x)-1$ predicted by the fundamental inequality (10). Here $x_1=-1,\ x_2=0,\ x_3 = 999,\ x_4 = 999999$.

Inequalities (10) provides the best upper and lower bounds on $Q(x)-1$ as a function of $L_R(x)$ depending on the given numbers of coeffcients of power series (4). The multipoint Padé approximant bounds given by (10) generalize all previous bounds reported in the literature, cf. [1,2,3,4]. From that point of view the general bounds (10) are new.

PARTICULAR CASES

From $Q(x) = 1+O(x),\ Q(x) = Q(-1)+O(x+1),\ Q(-1)\le 1$the elementary bounds follow.

$$(-1)^{H(x)}(1+x) \le (-1)^{H(x)}Q(x) \le (-1)^{H(x)},\ x\in[-1,\infty), \quad (11)$$

For $Q(x) = 1+\varphi_2 x+O(x^2),\ Q(x) = Q(-1)+O(x+1),\ Q(-1)\le 1$ the Wienner bounds result.

$$(-1)^{2H(x)}\left(1+\frac{\varphi_2 x}{1+\varphi_1 x}\right) \le (-1)^{2H(x)}Q(x) \le (-1)^{2H(x)}\left(1+\varphi_2 x\right),\ x\in[-1,\infty). \quad (12)$$

For $Q(x) = 1+\varphi_2 x+0.5\varphi_2\varphi_1 x^2+O(x^3),\ Q(x) = Q(-1)+O(x+1),\ Q(-1)\le 1$ the H–S bounds are obtaine

$$(-1)^{3H(x)}\left(1+\frac{\varphi_2 x+0.5\varphi_2 x^2}{1+0.5(1+\varphi_1)x}\right) \le (-1)^{3H(x)}Q(x) \le (-1)^{3H(x)}\left(1+\frac{\varphi_2 x}{1+0.5\varphi_1 x}\right),\ x\in[-1,\infty). \quad (13)$$

Here φ_1 and φ_2 denote the volume fractions of the first and second component of the two-phase composite.

CONCLUSIONS

By using special multipoint continued fraction technique, from several truncated power series we have derived the general inequalities for the multipoint Padé approximants to the effective transport coefficients (dielectric or diffusion constants, magnetic permeabilities, thermal or electrical conductivities) of two-phase media. Those inequalities are new and provide the best upper and lower bounds on $Q(x)$ over the entire class of rational functions. Note that they are obtained in a unified and coherent form as a function of $M_R(x)$ depending on given numbers of coeffcients of the power expansions of $Q(x)$ only. Moreover, they generalize all previously known relevant bounds, cf. [1,2,3,4].

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