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PARTICLE TRANSPORT BY A VORTEX SOLITON

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Summary Motions of fluid particles advected by a vortex soliton are studied. In the moving frame which makes the vortex soliton steady in space, particle motions are confined in a torus near the loop for a wide range of three parameters that characterize the shape and strength of the vortex soliton. To extract the essential mechanism of the transport properties, an ODE model is proposed, which is named the chopsticks model.

INTRODUCTION

Transport of fluid particles by an isolated vortex has been a fundamental problem which, particularly if a vortex moves steadily, provides a direct example of long-surviving advection of materials in fluids. The main objective of this paper is to demonstrate an example of finite volume transport by a 3D thin steady vortex tube, called vortex soliton named after Hasimoto [1]. The vortex soliton is one of few steady solutions for a vortex filament under the local induction approximation moving without changing its shape [2]. The original observation of vortex soliton in a rotating tank experiment implies that a vortex soliton is endowed with an ability to transport physical quantities, such as mass, kinetic energy, and linear and angular momenta, from a turbulent region to a laminar one [3][4]. Among others, mass is the most fundamental and important quantities in the transport phenomena of the vortex soliton, which in this paper is investigated by scrutinizing the flow field near the vortex soliton numerically [5] .

While the flow structure around the vortex soliton is quite convoluted, we could observe that some particles are confined in a torus near the loop of the vortex soliton. If the Poincaré section is taken for the torus, we see each particle inside the torus moves on an invariant surface which wraps a periodic orbit at the center of the torus. Similarity is also found for the structure of the torus with the KAM torus for non-integrable Hamiltonian systems. Figure 1 shows an example of the torus for the vortex soliton with a certain set of parameters for the shape and the strength of the soliton. The surface of the torus is weaved by a single particle trajectory in a moving frame which makes the vortex soliton steady. The edge of the torus marks a boundary in the flow field; outside particles are repelled into the flow while particles are continuously trapped inside. Thus the vortex soliton transports a finite volume of fluid in the form of a torus, and we are concerned with the transported volume as a function of the parameters for the properties of the vortex soliton.

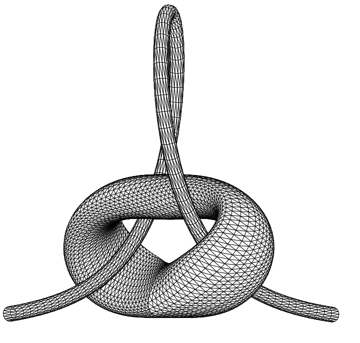

Figure 1. Torus with the vortex soliton.

FORMULATION

Hasimoto-vortex soliton is a solitary wave type solution of the Localized Induction Equation(LIE) for the motion of an isolated vortex filament,

$$\frac{\partial \mathbf{X}}{\partial t} = G\,\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial^2 \mathbf{X}}{\partial s^2}. \tag{1}$$

In the above equation $G = \frac{\Gamma}{4\pi}\log\left(\frac{L}{\epsilon}\right)$ is the self-induction constant where L and ϵ are two cut-off lengths, large and small, and Γ is the circulation. Usually G is absorbed by scaling the time variable, but we leave it here to have a specific time scale both for the vortex soliton and fluid particles. Following Hasimoto [1], the soliton solution for the above equation is given with two additional parameters $\nu > 0$ and τ, where ν is related with the curvature of a soliton $\kappa(s)$ by the formula $\kappa(s) = 2\nu\,\mathrm{sech}(\nu s)$ and τ is a constant torsion of a soliton. By denoting $\mathbf{X}(s,t) =^t [X_f(s,t), Y_f(s,t), Z_f(s,t)]$, we obtain

$$\begin{aligned} X_f(s,t) &= \frac{2\mu}{\nu}\,\mathrm{sech}\,\{\nu(s-2\tau Gt)\}\cos\left\{\tau(s-2\tau Gt)+(\nu^2+\tau^2)Gt\right\} \\ Y_f(s,t) &= \frac{2\mu}{\nu}\,\mathrm{sech}\,\{\nu(s-2\tau Gt)\}\sin\left\{\tau(s-2\tau Gt)+(\nu^2+\tau^2)Gt\right\} \\ Z_f(s,t) &= s-\frac{2\mu}{\nu}\tanh\{\nu(s-2\tau Gt)\} \end{aligned} \tag{2}$$

where $\mu = \nu^2/\left(\nu^2+\tau^2\right)$.

For the particle motion we assume that fluid particles are advected by the velocity induced by the vortex soliton through the Biot-Savart's integral taken over the centerline of the vortex tube. Under this assumptions, the equation of the motion of a fluid particle at $\mathbf{r}(t) = ^t [x(t), y(t), z(t)]$ in the frame which makes the vortex soliton steady is given as,

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\mathbf{x}'(s)\times(\mathbf{r}-\mathbf{x}(s))}{|\mathbf{r}-\mathbf{x}(s)|^3}\mathrm{d}s + (\nu^2+\tau^2)G\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2\tau G \end{bmatrix}, \quad (3)$$

where $\mathbf{x}(s) = \mathbf{X}(s, 0)$ is the coordinates of the vortex soliton steady in time, and the time variable t and G are re-scaled by the circulation. A certain congruent transformation is introduced to derive the second and the third terms in the right hand side to compensate the rotation about the z-axis and the translation in the z-direction. We use (3) as an autonomous dynamical system for the motion of particles around the vortex soliton. The solution to (3) with an initial condition $(x(0), y(0), z(0))$ provides a triplet $(x(t), y(t), z(t))$ which defines a trajectory in the 3D phase space.

NUMERICAL RESULTS

As a canonical tool to analyze trajectories in a phase space of higher dimension than two, the Poincaré section is quite well-known. Figure 2 shows the Poincaré sections for the trajectories of (3) with a various initial conditions for two different values of G, ((a): 0.1832, (b): 0.3113). The Poincaré plane is located at $y = 0$, and a dot is placed at a point on the plane every time when a trajectory passes the plane transversally at the point. On both the Poincaré sections, we can observe that there are two large islands of groups of nested curves (composed with continuous and broken lines) surrounded by small islands and scattered points. Each large island has a center which corresponds to a single periodic orbit in the original 3D space.

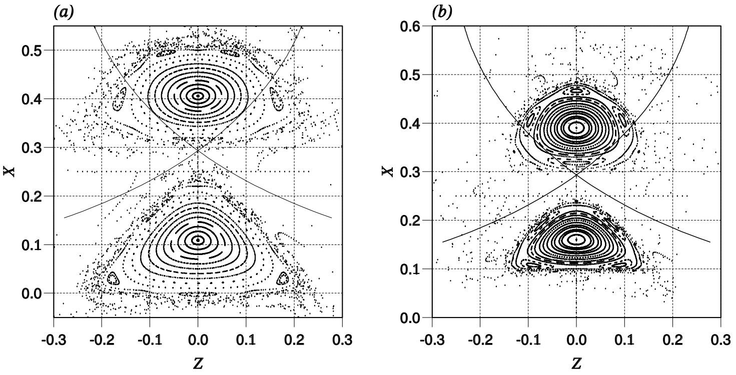

Figure 2. Poincaré sections for trajectories of (3) with various initial conditions. Poincaré plane is located at $y = 0$. The parameter values are $(\nu, \tau) = (1.924, 0.3827)$ and (a): $G = 0.1832$, (b): $G = 0.3113$.

Each nested curve around the center point, on the other hand, corresponds to an invariant surface of a torus around the periodic orbit. The scattered points outside the islands mean trajectories of particles repelled from the loop by the uniform translational flow in the moving frame. While (3) is not a Hamiltonian, very similar features are observed with the KAM torus for non-integrable Hamiltonian systems.

The most interesting and fundamental quantity about the transport by the vortex soliton is the fluid mass which can be estimated by calculating the volume surrounded by the outmost torus. Figure 3(a) shows the volume of the torus as a function of G for some pairs of ν and τ. As the Poincaré sections above suggest, a torus has a larger volume for smaller G for any combination of ν and τ. For a given G, a smaller value of τ gives always a larger values of the volume.

To understand and extract the essential mechanism of the formation of tori and the transport of volume, we propose a simplified toy model, in which the vortex soliton is replaced with two infinite straight line vortices in 3D space, and we call it the chopsticks model.

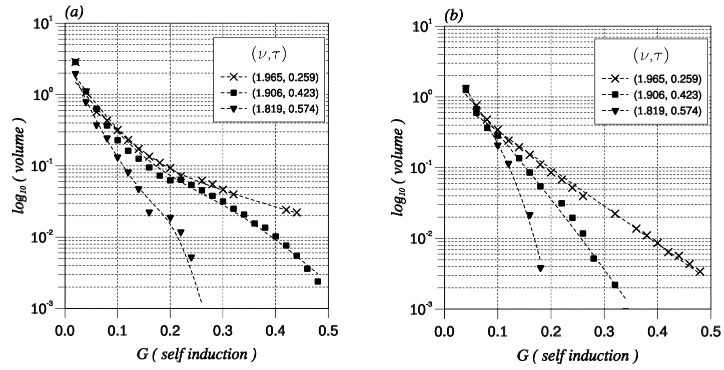

Figure 3. Comparison of the volume of the outmost torus as a function of G for three different combinations of ν and τ. (a) vortex soliton, (b) chopsticks model

This model successfully provides a similar torus structure and the dependence of the volume on the parameters. Figure 3(b) shows the volume obtained with the chopsticks model for the same parameter sets with Figure 3(a).

References

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