Intrinsic Formulation of Dynamics of Curvilinear Systems

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Introduction

In a lot of mechanical systems different models of curvilinear elements are used: string, beam, cable, rods,.... The formation of the general equations of such systems (large strains and displacements: translatory motions and rotations) remains a difficult problem which has been studied for a long time ([6], [2], [7] for example) We shall use the intrinsic differential calculation in a Lie group which has already been used in another cases (see [4] for example for an use in mechanics of systems of rigid bodies) to write the general and coordinate-free equations of any curvilinear system. The advantages are for example: the hexa-dimensional aspect, the generality of the calculations (without hypothesis of smallness of displacements or strains), the possibility of intrinsic limited expansion (linearisation for example), the possibility to point out precisely the difficulties (non linearities,...), the automatic generation of scalar equations.

1 Mathematic tools

$\mathcal{E}$ is the affine space of three dimensions of ordinary Euclidean geometry and E the associated vector space. $\mathbf{D}$ is the group of affine mappings A such that the linear part $\mathbf{A}$ is an element of the special orthogonal group $SO(E)$. $\mathcal{D}$ is the vector space of six dimensions of skew symmetric vector fields $X : \mathcal{E} \to E$ such that there exists $\boldsymbol{\omega}_X$ in E with the well-known following property: $\forall a, b \in \mathcal{E} \quad X(a) = X(b) + \boldsymbol{\omega}_X \wedge \overrightarrow{ba}$ and $\mathcal{D}$ shall be identified with the set of screws.
The Lie bracket is defined in $\mathcal{D}$ by $[X, Y](a) = \boldsymbol{\omega}_X \wedge Y(a) - \boldsymbol{\omega}_Y \wedge X(a)$ for all a in $\mathcal{E}$. Thus $\mathcal{D}$ is a Lie algebra which is isomorphic (and identified with) to the classical Lie algebra of $\mathbf{D}$.

2 Kinematics, Kinetics, Dynamics

The system is described as following. The reference configuration is similar to a distribution $\sigma \mapsto r(\sigma) = (A(\sigma); \vec{i}_1(\sigma), \vec{j}_1(\sigma), \vec{k}_1(\sigma))$ of affine frames where σ is the curvilinear abscissa of the curve $\sigma \mapsto A(\sigma)$. $A(\sigma)$ is **for example** the inertia center of the section of abscissa σ and $\vec{i}_1(\sigma), \vec{j}_1(\sigma), \vec{k}_1(\sigma)$ is a basis connected to the rigid section of abscissa σ (one may choose **for example** the Frenet frame of the curve $\sigma \mapsto A(\sigma)$). In the following, each rigid section and its associated frame shall be identified.
On each section $r(\sigma)$ and at each time t, is acting an unknowwn displacement $D(\sigma, t)$ such that $r(\sigma) \to s(\sigma, t) = D(\sigma, t) \bullet r(\sigma)$ where $s(\sigma, t) = (a(\sigma, t); \vec{i}_2(\sigma, t), \vec{j}_2(\sigma, t), \vec{k}_2(\sigma, t))$. Notice that no hypothesis of perpendicularity of the section with the curve $(\sigma, t) \mapsto a(\sigma, t)$ is made and $\bullet$ is the natural action of $\mathbf{D}$ on the set of affine frames (in mathematical terms it is a structure of fibered bundle).

The kinematics of the system is described by the following fields: the velocity $(\sigma, t) \mapsto v^c(\sigma, t) = \mathbf{D}(\sigma, t)^{-1} \circ \frac{\partial D(\sigma,t)}{\partial t}$ and the strain $(\sigma, t) \mapsto e^c(\sigma, t) = \mathbf{D}(\sigma, t)^{-1} \circ \frac{\partial D(\sigma,t)}{\partial \sigma}$ ($\sigma \mapsto e^c(\sigma, t)$ is unchanged by superposition of a rigid motion).
According to the chosen model, it is supposed that at each time t and on each section $r(\sigma)$ there are:

a distribution $(\sigma, t) \mapsto \mathcal{T}(\sigma, t)$ describing the massic external actions, a distribution $(\sigma, t) \mapsto \Theta(\sigma, t)$ describing the internal actions , action of the right-side on the left one (according to the abscissa σ), two concentrated massic strengths at the ends $\mathcal{T}_0(t), \mathcal{T}_l(t)$ (in some cases, one could consider a family $\mathcal{T}_k(t)$ at σ_k for $k = 1, \ldots, n$ which introduce a discontinuity of $\sigma \mapsto \Theta(\sigma, t)$ and we should use what follows on each $[\sigma_k, \sigma_{k+1}]$), a distribution $\sigma \mapsto H_r(\sigma)$ of operators of $\mathcal{D}$ describing the inertia strengths.

3 The equations

The equations of the system (Σ) are in a lagrangian expression (the functions of σ and of t are omitted):

$$\begin{aligned} \mathcal{T}^c &= \rho H_r(\dot{v}^c) + [v^c, \rho H_r(v^c)] - [e^c, \Theta^c] - \frac{\partial \Theta^c}{\partial \sigma} \\ \mathcal{T}_0^c = \Theta^c(0) \quad &, \quad \mathcal{T}_l^c = -\Theta^c(l) \end{aligned}$$

If some restrictions are done on the field $(\sigma, t) \mapsto D(\sigma, t)$ (planar motion, small displacement, perpendicularity of the section to the curve $(\sigma, t) \mapsto A(\sigma, t)$ (Kirchoff-love hypothesis) and so on), one comes back to the classical equations of beams, strings, cables and so on.
Moreover, choosing a basis in the Lie algebra $\mathcal{D}$ which may be canonically associated with a choice of an affine frame in the affine space, intrinsic relations become scalar ones. Linear operators become matrices, vectors become columns and the most part of the operations can be automatically done (with MAPLE for example). The main steps of a programm allowing to do it shall be presented in the fulle text and one of the expanded equation shall be given, its complexity justifying the intrinsic approach. The questions of constitutive laws and of linearized equations preserving the same intrinsic approach shall constitute future works and shall be suggested.

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