SHORT WIND WAVES AND SURFACE WIND DRIFT

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Summary The problem of short wind waves propagate on surface wind drift is considered here. The convergence of piecewise linear approximation for solving Rayleigh instability equation is proved. The method is numerically efficient and highly accurate. Applying this method, linear stability diagrams of various boundary flows are examined. It is also used to validate the applicability of other approximate approaches to the problem of propagation of short waves on wind drift shear. (*Acknowledgment:* This work is supported by the National Science Foundation. OCE0118028).

Under the assumption of small monochromatic surface waves on a steady horizontally uniform surface shear of an inviscid fluid, the governing equation becomes the well-known Rayleigh equation. The exact analytical solutions are found for a very limited number of current profiles. For arbitrary current profiles, approximate solutions are used[1,2]. The conditions for these approximations may be violated in the case of short wind waves on wind drift shear. As an alternative approach, the piecewise linear approximation (PLA) is explored.

THE CONVERGENCE OF THE PIECEWISE LINEAR APPROXIMATION

A continuous shear profile is approximated by constant vorticity layers. It is convenient to assume an uniform layer thickness, Δz. At the layer interfaces, the piecewise linear velocities are chosen to match the corresponding continuous shear velocities. Our PLA solution to the Rayleigh equation is an expansion series in powers of the layer thickness:

$$W(z)=W^{(0)}(z)+(\Delta z)^1W^{(1)}(z)+(\Delta z)^2W^{(2)}(z)+...$$

The boundary conditions at free surface and the bottom are the same for all the orders[2], so does the velocity continuity across the layer interfaces. At the zeroth order, the vorticity jump condition holds at layer interfaces. However, at the higher orders, there is an additional term from lower order residue as underlined below:

$$(c-\gamma U_i)[W^{(m)}{}'(z_i+\frac{\Delta z}{2})-W^{(m)}{}'(z_i-\frac{\Delta z}{2})+\underline{W^{(m-1)}}]=-\gamma[U'(z_i+\frac{\Delta z}{2})-U'(z_i-\frac{\Delta z}{2})]W^{(m)}(z_i),\qquad m>0,\ \ \Delta z\to 0$$

where γ is a non-dimensional scaling factor, c is the surface wave speed, z_i is the position of a layer interface, and U_i is the velocity value at z_i. $'$ denotes for vertical differentiation. The difference between finite difference of U' of PL velocity and that of $\mathcal{U}'(z)$ of the continuous profile is at an order of $(\Delta z)^2$:

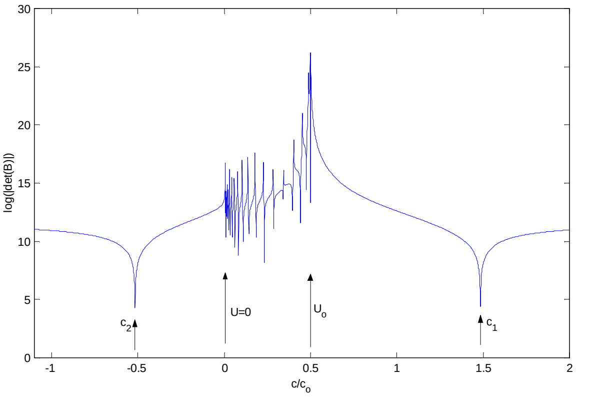
$$[U'(z_i+\frac{\Delta z}{2})-U'(z_i-\frac{\Delta z}{2})]=[\mathcal{U}'(z_i+\frac{\Delta z}{2})-\mathcal{U}'(z_i-\frac{\Delta z}{2})]+\mathcal{O}\left(\Delta z^2\right)$$

If $W^{(m)}{}'(z)$ is continuous at $z=z_i$ as $\Delta z\to 0$, and the second derivative, $W^{(m)}{}''(z)$, at the layer interfaces also becomes well-defined at the limit, then, we can show:

$$(c-\gamma\mathcal{U})\left(W^{(0)}{}''+W^{(0)}\right)+\gamma\mathcal{U}''W^{(0)}=0\quad z=z_i$$

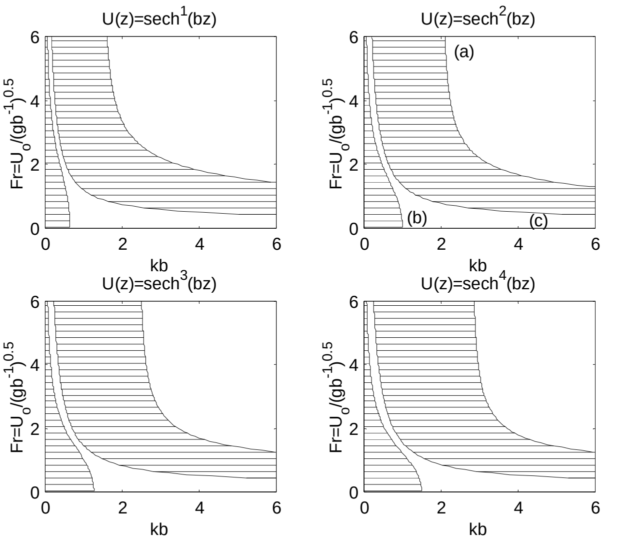
Although, the higher order terms become insignificant as $\Delta z\to 0$, they are necessary only to uphold the convergence. For an eigen value problem, if c is an eigen value admitted for both of the continuous and PL systems, then, the solution is unique subject to the same surface and bottom boundary conditions.

The PLA solution can be found by solving a homogenous algebra equation: $[B]W=0$, and the dispersion relation is defined by $\det[B]=0$. The left figure shows the logarithm of absolute value of $\det[B]$ evaluated at real values of wave speed c for a $\operatorname{sech}^2(z)$ surface shear profile. In the figure, $c=\pm 1$ correspond the wave speeds of linear waves in still water (normalized). The roots of the dispersion relation are at the dips in the figure where the determinant changes sigh when it crosses zero. The shear current is in the range of [0 .5]. Normally, there are two roots that are associated with nature surface waves, c_1, c_2, with c_1 travels along and c_2 travels against the current. The other roots are those modes whose phase speeds equal the current speed at some depth, $c-\gamma U(z_c)=0$.



$U''=0$ everywhere except at the layer interfaces for PLA, while there are normaly only a limited number of inflection points for a realistic shear profile. It has been shown by Yih[3] that neutral solutions are not allowed at non-inflection points for most of the boundary flows. Many neutral solutions from the PLA are artificial because of the real $\mathcal{U}''(z_c)\neq 0$. One has to be very cautious in applying the PLA in the range of $U_{\min}\le c\le U_{\max}$.

THE NEUTRAL MODES OF CERTAIN SURFACE SHEAR FLOWS



The instability of a horizontal shear flow with a free surface has important applications to the study of surface wakes of ships or near water surface bodies; to the stability of the crest of a spilling wave breaker; and to wind-drift currents. There are only a few cases that analytic solutions are found, thus, it largely depends on elaborated numerical calculations to solve for realistic surface shear profiles.

A well-studied case is the linear instability of a shear flow with a velocity profile in the form of $\mathcal{U}=\mathcal{U}_o \mathrm{sech}^2(bz)$. This profile has been numerically studied [4,5] to fit experimental data of the surface shear flow in a wake of a hydrofoil. For the linear instability analysis, the neutral modes form the stability boundaries in the wavenumber and Froude number space (k, Fr). In addition, an analytic solution is found for neutral modes at the inflection points[6]. Knowing the valid eigen value c_s for the continuous current profiles, the corresponding wavenumber and waveform can be very precisely calculated by PLA. In the figure above, the linear stability diagram for surface gravity waves are shown for the family profiles of $\mathcal{U}=\mathcal{U}_o \mathrm{sech}^n(bz)$. For $n=2$, PLA accurately matches with the analytic calculation[6]. Here the Froude number is defined as $Fr=\mathcal{U}_o\big/\sqrt{gD}, D=b^{-1}$. The curves (a) and (b) are the lico of the neutral solution associated with velocity $c=\gamma \mathrm{sech}^n\left[\tanh^{-1}\left(\sqrt{1/n}\right)\right]$, and the curve (c) the locus of the neutral solution with $c=0$. The shaded regions are the unstable regions. The linear stability diagrams for other boundary shear flows, such as $\mathcal{U}=\mathcal{U}_o(1-\tanh(b(y/D)^2)$, exponential, error function, and Blasius profile, are also examined with and without including surface tension.

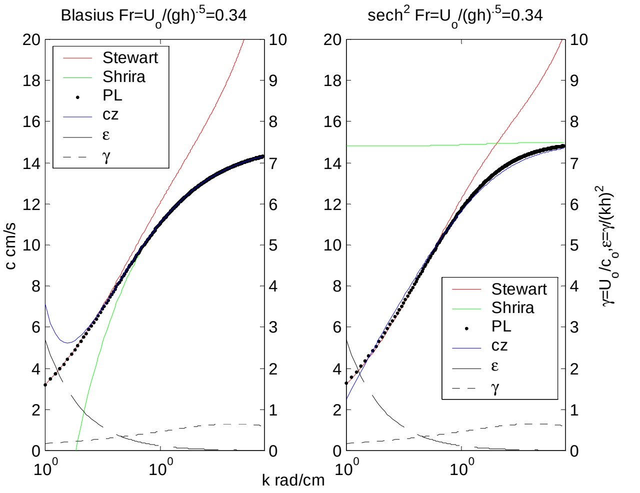
PROPAGATION OF SHORT WIND WAVES ON WIND DRIFT

The mechanism that modifies the propagation of surface waves by currents is well understood. However, the applications are so far still limited to the simple velocity profiles. First of all, it is not an easy task to measure the profile of the surface currents in most cases. Secondly, our calculations on propagation speed rely heavily on the perturbation approximations which may not hold for short waves. With PLA, we are able to calculate the wave propagation speed accurately for a given realistic surface flow profile, thereby, to test perturbation approximations.

Also, an implicit expression for the wave speed can be found by integrate the equation from the bottom to the surface:

$$c=U_o-c_o\frac{k}{\kappa}\left[\frac{U_o'}{2c_ok}\pm\sqrt{\frac{\kappa}{k}+\left(\frac{U_o'}{2c_ok}\right)^2}\right] \quad \text{where } \kappa=\int_{-\infty}^{0}dz\left[\left(\frac{W'}{W(0)}\right)^2+\left(\frac{W}{W(0)}\right)^2\left(k^2+\frac{U''}{U-c}\right)\right].$$

A suitable trail function is $W \doteq W(0)e^{kz}$, which yields: $\kappa=k+\int_{-\infty}^{0}dz\left(e^{2kz}\frac{U''}{U-c}\right)$. Figure below is an example of different shear profiles with their Froude numbers close to the case of 5 m/s wind speed estimated from Wu's laboratory measurements[7]. Stewart approximation holds well for long waves while Shrira's approximation is good for high wavenumbers (the first order approximation only). Our approximation fits well in a wider wavenumber range. However, a better physical justification is needed.



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