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ACCELERATION WAVEFRONTS IN RANDOM MEDIA

Martin Ostoja-Starzewski*, Jerzy Trebicki **
*Department of Mechanical Engineering, McGill University, Montreal, Que. H3A 2K6, Canada
**Polish Academy of Sciences, IPPT PAN, Swietokrzyska 21, PL 00-049 Warszawa, Poland

Summary In contradsitinction to deterministic continuum mechanics, we consider wavefronts whose thickness is smaller than the Representative Volume Element (RVE) size. As a result, the wavefront is an object more appropriately analyzed as a Statistical Volume Element (SVE) rather than an RVE, and therefore to be treated via a stochastic, rather than a deterministic, dynamical system.

INTRODUCTION

In conventional deterministic continuum mechanics, wavefronts are commonly modeled as singularity surfaces traveling in homogeneous media. Although a wavefront's thickness is very thin - mathematically infinitesimal, indeed - the ensuing analyses assume that the Representative Volume Element (RVE) size is even smaller (!) and no consideration of material spatial randomness is made. Whereby a typical grains' size $\langle d \rangle$ is supposed to tend to zero relative to the wavefront's thickness L. In contradistinction to that classical approach, just like in our previous studies [1, 2] we consider the wavefront as an object much more appropriately analyzed as a Statistical Volume Element (SVE). Our focus is primarily on acceleration wavefronts, and there are two entirely new aspects considered in the present study. One is the explicit consideration of spatial randomness and various cross-correlations of the instantaneous modulus, the dissipation coefficient, the instantaneous second-order tangent modulus, and the reference state mass density. The second new facet is the coupling of these four random fields to the wavefront amplitude: as the amplitude grows, the wavefront gets thinner tending to a shock, and thus the material heterogeneity shows up as an ever `stronger' random field.

BACKGROUND

Acceleration waves are generally governed by the Bernoulli equation [e.g. 3]

$$\frac{d\alpha}{dx} = -\mu\alpha + \beta\alpha^2 \tag{1}$$

Here x is position, α is jump in particle acceleration, while the coefficients μ and β represent, respectively, two effects: dissipation and elastic non-linearity. The most interesting aspect of acceleration waves uncovered through this equation is that, due to the competition between these two effects, there is a possibility of blow-up, and hence, of shock formation in a finite distance x_∞, providing the initial amplitude α_0 exceeds a *critical amplitude* α_c; x_∞ is called the *distance to blow-up*.
The explicit formulas for the dissipation coefficient, the nonlinear amplification coefficient, and the velocity of acceleration wave are [3]

$$\mu = -\frac{G_0'}{2G_0} \quad , \quad \beta = -\frac{\widetilde{E}_0}{2G_0}\sqrt{\frac{\rho_R}{G_0}} \quad , \quad c = \sqrt{\frac{G_0}{\rho_R}} \tag{2}$$

where G_0 is called the instantaneous modulus, G_0' is the coefficient responsible for dissipation, is $\widetilde{E}_0$ the instantaneous second-order tangent modulus, and ρ_R is the mass density in the reference state. The dynamical system (1) is driven by these four basic material coefficients, and, given their randomness, both α_c and x_∞ are random processes.

BERNOULI EQUATION PERTURBED BY VECTOR RANDOM PROCESS

Wavefront evolution is governed by a stochastic differential equation

$$\frac{d\alpha}{dx} = \frac{G_0'}{2G_0}\alpha - \frac{\widetilde{E}_0}{2G_0}\sqrt{\frac{\rho_R}{G_0}}\alpha^2 \tag{3}$$

driven by a four-component random process $\Theta_x = [G_0, G_0', \widetilde{E}_0, \rho_R]$ with

$$\begin{aligned} G_0 &= \langle G_0 \rangle + S_1 X_1(x,\omega) > 0 \ , \quad G_0' = \langle G_0' \rangle + S_2 X_2(x,\omega) \le 0 \\ \widetilde{E}_0 &= \langle \widetilde{E}_0 \rangle + S_3 X_3(x,\omega) \le 0 \ , \quad \rho_R = \langle \rho_R \rangle + S_4 X_4(x,\omega) > 0 \end{aligned} \tag{4}$$

where $\langle \cdot \rangle$ denotes the mean value, $\mathbf{X}(x,\omega) = [X_i(x,\omega)]_{i=1}^4$ is a standard Gaussian vector random process with zero mean value $\langle \mathbf{X}(x,\omega) \rangle = 0$ ($\langle X_i(x,\omega) \rangle = 0$), variances $\operatorname{var} X_i = 1$, and positive defined correlation matrix $K_{ij}(\Delta x) = \langle X_i(x,\omega) X_j(x+\Delta x,\omega) \rangle$, $\omega \in \Omega$ where Ω is an element of the probability space $\{\Omega, F, P\}$. We assume that parameters S_i satisfy condition $S_i << 1.0$. Upon transformations, we may write:

$$\mu(x,\omega) = -\frac{\langle G_0^{'} \rangle + S_2 X_2(x,\omega)}{2[\langle G_0 \rangle + S_1 X_1(x,\omega)]} \quad , \quad \beta(x,\omega) = -\frac{[\langle \tilde{E}_0 \rangle + S_3 X_3(x,\omega)]\sqrt{\langle \rho_R \rangle + S_4 X_4(x,\omega)}}{2[\langle G_0 \rangle + S_1 X_1(x,\omega)]^{3/2}} \tag{5}$$

so that the 1-*D* statistics of $\mu(x,\omega)$ and $\beta(x,\omega)$, and of α_c and x_∞, may be determined from $[X_i(x,\omega)]_{i=1}^4$. The first case which we investigate in depth is that of all $X_i(x)$'s being a stationary Ornstein-Uhlenbeck (O-U) process $U(x,\omega)$ with a correlation function $K_U(\Delta x) = \sigma_U^2 \exp(-a|\Delta x|)$ where $\sigma_U^2 = 1$, whereby the dynamical system is

$$\frac{d\alpha}{dx} = -\mu(U(x,\omega))\alpha + \beta(U(x,\omega))\alpha^2 \ , \quad dU = -aUdx + \sqrt{2a}\,dW_x(\omega) \tag{6}$$

with $W_x(\omega)$ being the standard Wiener process; $\alpha(x=0) = \alpha_0$. As an example, Fig. 1 shows the probability densities of x_∞ at various values of parameter *a*, all obtained via the method of Winterstein (e.g. [1]) for first four statistical moments of x_∞ obtained as result of Monte Carlo simulations of equations (6) where O-U process starts from its stationary state. The cases when processes $[X_i(x,\omega)]_{i=1}^4$ are independent or slightly correlated will be also reported during the presentation.

COUPLING OF THE AMPLITUDE OF AN ACCELERATION WAVE WITH MEDIUM'S SPATIAL RANDOMNESS

We now modify the original Bernoulli equation (6) by introducing the coupling between the processes $\alpha(x,\omega)$ and $\mu(x,\omega)$, $\beta(x,\omega)$ in the following way

$$\begin{aligned} &\frac{d\alpha}{dx} = -\mu(U)\alpha + \beta(U)\alpha^2 \ , \quad dU = -a(\alpha)Udx + \sigma_U(\alpha)\sqrt{2a(\alpha)}\,dW_x(\omega) \\ &\frac{d\sigma_U}{dx} = C_1\alpha^{m_1} \ , \quad \frac{da}{dx} = C_2\alpha^{m_2} \ , \quad \sigma_U(x_0) = 1 \ , \quad a(x_0) = a_0 \ , \quad C_1, C_2, m_1, m_2 > 0 \end{aligned} \tag{7}$$

Thus, as α grows, so does the variance of the driving random fields μ and β, and decreases their correlation length. Of course, in the particular case of the Ornstein-Uhlenbeck process, $r=1/a$. Figure 2 shows comparison of distribution functions of x_∞ (for small probabilities) when there is no coupling vis-à-vis when there is coupling as modelled here. It is seen that the introduction of coupling causes growth of the probability of blow-up by up to factor two for the considered values of parameters. Related effects on α_c will also be reported orally. Our presentation will end with an extension of the concept of SVE to shock waves in random media and with the setup of a corresponding stochastic dynamical system governing their evolution.

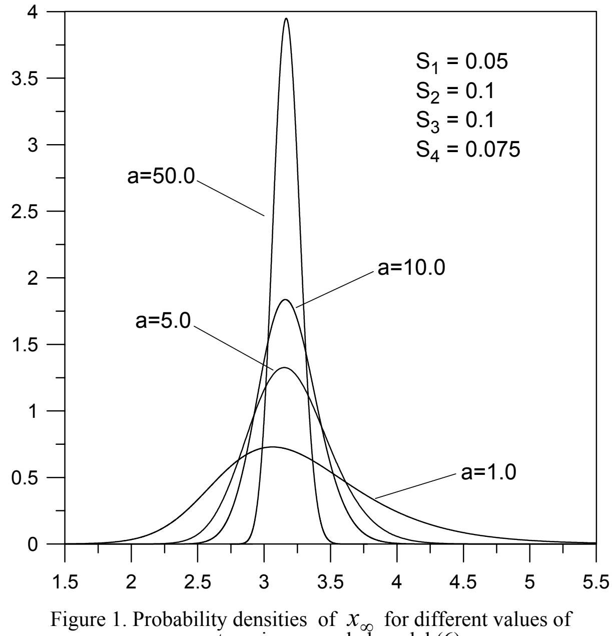

Figure 1. Probability densities of x_∞ for different values of parameter *a* in uncoupled model (6).

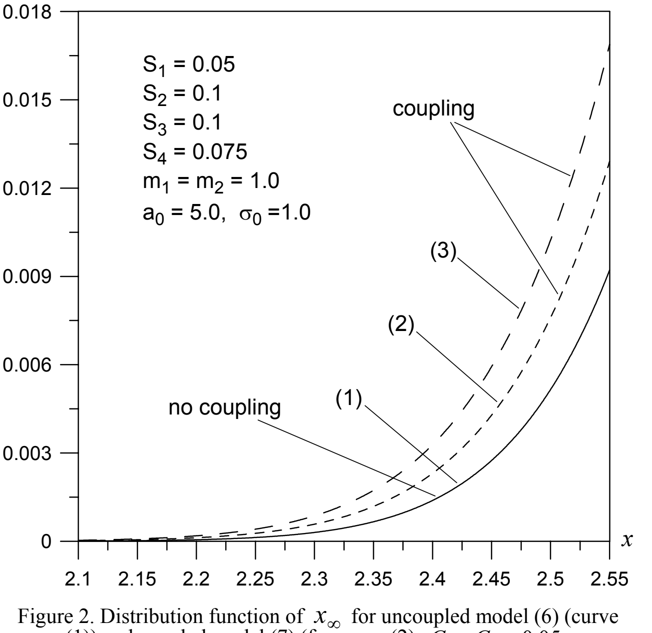

Figure 2. Distribution function of x_∞ for uncoupled model (6) (curve (1)) and coupled model (7) (for curve (2): $C_1 = C_2 = 0.05$, and for curve (3): $C_1 = C_2 = 0.1$)

References

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