ON THE SOURCE OF SINGULARITIES IN MECHANICS

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Summary Stress singularities occur in both fluid and solid mechanics. Such singularities may be viewed as breakdowns in the modeling which produces them. To begin to improve this modeling, the source of these singularities needs to be identified. Here this is demonstrated to be infinite stiffnesses in underlying intermolecular laws. When finite stiffnesses are introduced instead, finite stresses result for a diverse variety of erstwhile singular configurations.

INTRODUCTION

In fluid and solid mechanics, stress or pressure singularities appear at sharp corners, step modifications to boundary conditions, and abrupt changes in continuum properties. Such discontinuity singularities abound in solid mechanics (e.g., [1]). Though perhaps not as widely recognized, they also occur quite frequently in fluid mechanics (e.g., [2,3]).

The stresses attending discontinuity singularities qualitatively reflect physical stress concentrations, but are quantitatively physically inapplicable. While there has been some success in interpreting these physically wayward fields, in general they remain a serious impediment to meaningful physical inferences. Consequently it is of interest to try and improve the modeling that occasions discontinuity singularities, and thereby eliminate them.

One longstanding suggestion to this end (Kelvin circa 1860) is to remove the discontinuity associated with the singularity by, say, rounding corners. For the example of a cracked solid, certainly there are no singular stresses with any root radius that is greater than zero. However, for a radius that is zero we get the physical absurdity of infinite stresses. This raises the question of just how physically relevant are the finite but extremely large stresses that result from extremely small radii. Moreover, crack tips can have such radii, so the question is not moot. And similar concerns apply to rounding other sharp corners.

Other suggestions entail modifications to the governing field equations. In general such modifications do not really remove singularities. This is shown in the literature for solid mechanics (for plasticity, viscoelasticity, large strains and displacements, couple stresses, and inertial effects), and to some extent for fluid mechanics (for viscosity, compressibility, and inertial effects). What needs to be changed to remove discontinuity singularities in both field theories is the boundary conditions. Specifically, infinite stiffnesses in underlying intermolecular laws need to be removed. We demonstrate this next.

SAMPLE SINGULAR/NONSINGULAR CONFIGURATIONS

We begin with a simple one-dimensional example from particle dynamics. This concerns a sphere of mass m and velocity V normally impacting a relatively rigid wall. Then the maximum contact force F when the sphere comes to rest given by

$$F = V\left(km\right)^{1/2},\ 8\left(V^6k^2m^3\right)^{1/5}/7. \qquad (1)$$

The first expression for F stems from taking a linear force-displacement relation with stiffness k. The second stems from taking a nonlinear relation corresponding to Hertzian contact of a rubber (incompressible) ball with $k=5E\sqrt{R}$. What (1) shows is that, in either case, we have finite contact forces and stresses if k is finite. We draw on this appreciation in our next example.

For our second example, we consider the steady flow without separation of an ideal incompressible fluid of density ρ over an impervious step. The fluid has an upstream velocity of V. In terms of r, θ coordinates, locally it flows through a region bounded by $r>0, |\theta|<3\pi/4$. Traditional boundary conditions take the velocity normal to the step to be zero, and lead to singularities in both velocity v and stress σ fields, viz.,

$$v_\theta = 0 \,@\, \theta = \pm 3\pi/4 \Rightarrow v = \mathrm{ord}\left(V\left(l/r\right)^{1/3}\right) + O\left(r^{1/3}\right),\ \sigma = \mathrm{ord}\left(\rho V^2\left(l/r\right)^{2/3}\right) + O(1), \qquad (2)$$

as $r\to 0$, where l is a characterizing length. Similar singular fields participate for other proud step corners.

Now instead treat the molecules comprising the fluid as spheres impacting a wall as in our original example. That is, allow for the deformation of these molecules as they impact. One simple means of doing this is to replace the traditional boundary conditions in (2) with a linear relationship between the normal stress and velocity, then nonsingular v and σ result, viz.,

$$\sigma_\theta = \mp k v_\theta \,@\, \theta = \pm 3\pi/4 \Rightarrow v = \mathrm{ord}\left(\rho^{-1}k\left(\left(1+2\rho^2V^2k^{-2}\right)^{1/2}+1\right)\right) + o(1), \sigma = \mathrm{ord}\left(kv\right) + o(1), \qquad (3)$$

as $r\to 0$. Thus we have no singularity for k finite. Similar results can be obtained for other proud corners. Results of like character can also be obtained if the first of (3) is replaced with a relation reflecting something akin to Hertzian contact.

If we revisit our step flow example with viscosity included, then either for Stokes flow or for the full Navier Stokes equations, the analogues of the boundary conditions in (2),(3) lead to

$$\sigma = O\left(r^{-0.46}\right), O\left(r^{-0.09}\right),\ \sigma = \mathrm{ord}\left(k\right) + O(1), \qquad (4)$$

as $r\to 0$, respectively, with bounded v for both. That is, once deformation is admitted with finite k, singularities are removed.

Further examples in fluid mechanics of configurations that can be rendered nonsingular with a like approach to the foregoing are: Taylor's scraping problem for an incompressible viscous fluid (singularity given in [2]), and Reynolds equation for a compressible air bearing when contact occurs (singular and nonsingular fields indicated in [3]).

Turning to solid mechanics, we first consider a stress-free reentrant corner under torsion. The elastic material comprising the corner locally occupies the same region as the fluid earlier ($r>0, |\theta|<3\pi/4$). For antisymmetric response, the out-of-plane displacement is traditionally set to zero on the bisector. This together with zero shear on a face lead to antiplane shear stresses τ that are singular, viz.,

$$w=0 @ \theta=0, \tau_{\theta z}=0 @ \theta=3\pi/4 \Rightarrow \tau=\operatorname{ord}\left(G\bar{W}\left(l/r\right)^{1/3}\right)+O(r), \tag{5}$$

as $r\to 0$, where G is the shear modulus, $\bar{W}$ a dimensionless representative displacement. Similar singular stresses participate for other reentrant corners.

Now for a sharp corner, consistency requires that cohesive laws must act between the flanks as the corner is approached to within molecular distances because these are the laws giving rise to the constitutive relations within the bulk of the continuum. With this appreciation it becomes apparent that, when separating the corner into two halves as in (5), we need to formulate problems for the molecules comprising the boundaries of the two halves. These molecular boundaries can be displaced with respect to one another along the bisector, so that the first of (5) only holds for their average displacement. Instead, therefore, we replace the first of (5) with a cohesive stress-separation law, then nonsingular shear components result, viz.,

$$\tau_{\theta z}=2kw @ \theta=0^{+} \Rightarrow \begin{Bmatrix}\tau_{rz}\\ \tau_{\theta z}\end{Bmatrix}=kW\left[\sin\theta\begin{Bmatrix}-\\ +\end{Bmatrix}\cos\theta-2\frac{kr}{G}\begin{Bmatrix}\sin 2\theta\\ \cos 2\theta\end{Bmatrix}\right]+O\left(r^{1/3}\right), \tag{6}$$

as $r\to 0$. Thus no singularity for finite k. Similar results can be obtained for other reentrant corners.

As a second example, we consider a composite wedge under torsion. The wedge is comprised of an elastic sector occupying $0<\theta<3\pi/4$ and a rigid sector occupying $3\pi/4<\theta<5\pi/4$, and has stress-free exterior faces. With traditional perfect matching assumed on the interface, the boundary conditions for the elastic sector are as in (5) but with θ values reversed, and lead to singular shears also as in (5). With an adhesive law instead on the interface, the first of (5) is replaced with the boundary condition in (6) but on $\theta=3\pi/4$ with $2k\to -k$, and nonsingular shears as in (6) result with $\theta\to 3\pi/4-\theta$. Thus no singularity for finite k. Similar results can be obtained for other bimaterial wedges.

If we revisit our reentrant corner example within the context of plane elasticity, singularities as in (4) result with traditional symmetry, antisymmetry conditions (stress-free conditions in plane elasticity are analogous to zero-velocity conditions in Stokes flow). If instead we introduce cohesive laws with finite k, nonsingular stresses as in (4) result. And such removal of singularities holds in general throughout plane elasticity (requisite asymptotic analysis follows [4]). That is, with the appropriate introduction of cohesive/adhesive laws with finite k, all the singularities in [1] can be removed.

There is one caution of note in effecting this removal of singularities throughout two-dimensional elasticity. This is for cracks when stiffnesses on crack flanks are reduced from those ahead of crack tips to model stress concentrations at tips. If shear stresses are involved, such reductions need to be continuous, otherwise log singularities can be shown to be induced. With this caveat, however, even the interface crack can be rendered completely free of stress singularities.

As a final example we consider the now classical three-dimensional (3D) problem of a crack intersecting the surface of a half-space. Traditional symmetry and antisymmetry conditions ahead of the crack set one out-of-crack-plane displacement and two in-crack-plane displacements to zero, respectively. Thus, in effect, symmetry conditions imply one infinite stiffness in a cohesive law while antisymmetry imply two. Hence symmetry conditions lead to one singularity, antisymmetry to two for the 3D crack problem (see [5]). Replacing these traditional conditions with cohesive laws can be shown (via the alternating method) to remove all three singularities.

For some of the preceding two-dimensional examples with traditional boundary conditions, asymptotic expansions can be shown to be complete only when the singular eigenfunctions are included. This all but guarantees the participation of these singular fields in corresponding global problems. Irrespective of completeness, participation of such singularities has been confirmed for a number of global problems with diverging numerical analyses (space limitations prevent summarizing results here). For some of the two-dimensional examples with intermolecular laws in boundary conditions, asymptotic expansions can be shown to be complete without any singular eigenfunctions. Irrespective of completeness, the nonparticipation of singularities has been confirmed for a number of global problems with converging numerical analyses.

Last we observe that, in all the local configurations with intermolecular laws in their boundary conditions, taking $k\to\infty$ makes these conditions revert to traditional ones with their singularities. This is therefore consistent with the identification of infinite stiffnesses as sources of discontinuity singularities in mechanics.

CONCLUSIONS

There are a wide variety of stress singularities at discontinuities in fluid and solid mechanics. Traditional boundary conditions for these singularities effectively contain infinite stiffnesses in underlying intermolecular laws. When finite continuous stiffnesses are used instead, finite stresses result, even with the original discontinuity still present. Thus it is infinite stiffnesses that are the source of singularities rather than their associated discontinuities themselves. This recognition offers the potential for replacing singular stresses with physically meaningful stresses throughout mechanics.

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