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AMPLIFICATION OF NONLINEAR DISTURBANCES ON A FALLING LIQUID SHEET

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Summary We analytically examine behavior of a liquid sheet falling under the action of gravity when liquid viscosity and inertia of surrounding fluids are ignored. Analysis is made based on a set of nonlinear equations for the sheet derived in the gravitational field under the membrane approximation. We numerically find a particular boundary condition for the steady flow whose velocity increases and thickness decreases monotonically as it goes downstream. Weakly nonlinear analysis for unsteady modulational waves shows that only antisymmetric mode of disturbances propagating downstream is amplified and otherwise decayed. Numerical analysis shows that the symmetric mode is locally induced on the antisymmetric mode and it is expected that this induced mode leads to the breakup of the sheet.

INTRODUCTION

Investigations of a liquid sheet are of great importance in scientific and technological applications. In particular, distortion and disintegration of a falling liquid sheet under the action of gravity are important in related problems to the coating technology and it is evident that nonlinearity plays an important role in such behavior of the sheet. As is well known, there can exist two different modes of disturbances on a planar sheet, that is, the antisymmetric (or sinuous) and symmetric (or bulge) modes, and both modes are linearly stable if inertia of surrounding fluids is neglected. On the other hand, for the sheet falling in the gravitational field, it is shown that the antisymmetric mode of infinitesimal disturbances propagating upstream is amplified due to the viscosity. Since the viscosity does not affect on the antisymmetric mode for such a thin sheet, however, this amplification disappears when the sheet thickness becomes thin. In this paper, we show that another amplification is possible if modulational waves are considered on the falling sheet even if the sheet thickness is sufficiently thin. In addition to this, we also show that the nonlinear behavior of the amplified disturbances leads to the breakup of the sheet.

SHEET EQUATIONS AND STEADY SOLUTIONS

We consider a falling liquid sheet in the (x, y) coordinate system as shown in Fig.1, where g denotes the gravitational acceleration, (u, v) the velocity components and p the excess pressure. In addition, the half thickness a and the sheet center plane η are given as $a = (h_+ - h_-)/2$ and $\eta = (h_+ + h_-)/2$ when the sheet boundaries are specified by $y = h_\pm(x, t)$. All dependent variables in the above are normalized by the thickness A_0 and the velocity U_0 at the reference point $x = 0$. According to the membrane approximation where the sheet thickness is so thin that internal structure of the sheet can be ignored, the original full nonlinear two-dimensional equations can be reduced to the following one-dimensional equations:

$$a_t = -(au)_x, \quad \eta_t = v - u\eta_x, \quad v_t = -uv_x - \Delta P/(2a\,\mathrm{We}),$$
$$u_t = -uu_x - [P_x - (\Delta P/2a)\eta_x]/\,\mathrm{We} + 1/(2\,\mathrm{Fr}^2), \qquad (1)$$

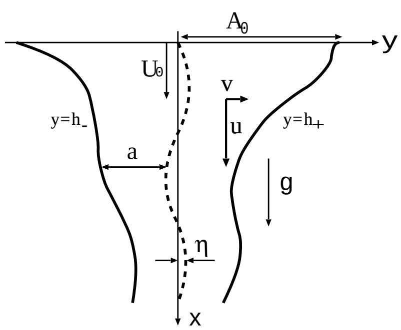


Fig.1 Schematic of a falling liquid sheet.

where u, v and p are functions of x and t under the approximation, and Fr and We, respectively, denote the Frude number and the Weber number. In the above representations, the mean pressure $P = (P_+ + P_-)/2$ and the pressure difference $\Delta P = P_+ - P_-$ are introduced in terms of $P_\pm = \mp(\eta_{xx} \pm a_{xx})[1 + (\eta_x \pm a_x)^2]^{-3/2}$ on the sheet surfaces $y = h_\pm$. For the steady state, we can assume that only symmetric mode can exist, and so that $\eta = v = 0$ and $\Delta P = 0$ in eqs.(1). Thus, we obtain the following set of equations with respect to $\overline{u}(x)$ and $\overline{a}(x)$ for the steady state:

$$\overline{au} = 1, \quad \overline{a}^{-3}\overline{a}_x + [\overline{a}_{xx}(1 + \overline{a}_x^2)^{-3/2}]_x/\,\mathrm{We} + 1/(2\,\mathrm{Fr}^2) = 0. \qquad (2)$$

The solutions of eqs.(2) are approximately obtained to be $\overline{a} = (x/\,\mathrm{Fr}^2 + 1)^{-1/2}$ for sufficiently large We , while $\overline{a} = (1 + x/\,\mathrm{Fr}^2)^{-1/2} - 3(4\,\mathrm{We}\,\mathrm{Fr}^4)^{-1}[(x/\,\mathrm{Fr}^2 + 1)^{-4} - (x/\,\mathrm{Fr}^2 + 1)^{-3/2}]$ for sufficiently large Fr. On the other hand, eqs.(2) can be numerically integrated for boundary values of $\overline{a}_x$ and $\overline{a}_{xx}$ when $\overline{a}$ is given. We note, however, that these values for the solutions representing the steady flow, whose velocity increases and width decreases monotonically as it goes downstream, should be particularly given for individual values of We and Fr. For example, $\overline{a}_x(0) = -0.049821\cdots$ and $\overline{a}_{xx}(0) = 0.007387\cdots$ when We=10, Fr=$\sqrt{10}$ and $\overline{a}(0) = 1$.

MODULATIONAL INSTABILITY

In order to see unsteady behavior of disturbances on the falling sheet, we assume that rapidly oscillating unsteady disturbances are superposed on the gradually varying steady flow and such solutions can be written as:

$$a = \overline{a}(x_2) + \tilde{a}(x_0, x_1, x_2, \cdots, t_0, t_1, t_2 \cdots), \qquad \eta = \tilde{\eta}(x_0, x_1, x_2, \cdots, t_0, t_1, t_2 \cdots),$$
$$u = \bar{u}(x_2) + \tilde{u}(x_0, x_1, x_2, \cdots, t_0, t_1, t_2 \cdots), \qquad v = \tilde{v}(x_0, x_1, x_2, \cdots, t_0, t_1, t_2 \cdots), \tag{3}$$

where the multiple scales $x_j = \epsilon^j x$ and $t_j = \epsilon^j t$ $(j = 0, 1, 2, \cdots)$ are introduced in terms of a small parameter ϵ, in order to differentiate between the phenomena with shorter scales and longer scales. For the steady flow, we choose the approximate solution $\bar{a}(x_2) = (x_2/\,\mathrm{Fr}_0^2 + 1)^{-1/2}$ with longer scale x_2 when $\mathrm{Fr} = \epsilon^{-1}\,\mathrm{Fr}_0$ and $\mathrm{We} \gg 1$, while we set the unsteady disturbance to be:

$$\tilde{a} = \Sigma_{n=1}\epsilon^n A_n(\theta, x_1, x_2, \cdots, t_1, t_2, \cdots) + C.C., \quad \tilde{\eta} = \Sigma_{n=1}\epsilon^n E_n(\theta, x_1, x_2, \cdots, t_1, t_2, \cdots) + C.C., \quad \cdots \tag{4}$$

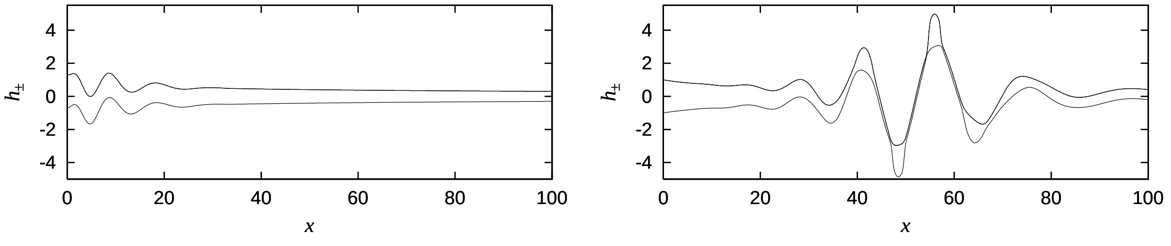
where A_n and E_n are complex amplitudes and $C.C.$ denotes complex conjugate, while we have assumed that the wave number k and angular frequency ω in the rapidly oscillating phase $\theta = kx_0 - \omega t_0$ are functions of x_2 corresponding to $\bar{u}$ and $\bar{a}$. Making use of the derivative expansions $\partial/\partial x = k\partial/\partial\theta + \Sigma_{n=1}\epsilon^n\partial/\partial x_n$ and $\partial/\partial t = -\omega\partial/\partial\theta + \Sigma_{n=1}\epsilon^n\partial/\partial t_n$ into eqs.(1), from the non-secular condition we finally obtain the following amplitude equations for the symmetric and antisymmetric modes, respectively:

$$\mathrm{i}(A_{1t_2} + V_g A_{1x_2}) + P_s A_{1x_1x_1} = Q_s A_1|A_1|^2 + \mathrm{i}\,T_s A_1, \quad \mathrm{i}(E_{1t_2} + V_g E_{1x_2}) = Q_a E_1|E_1|^2 + \mathrm{i}\,T_a E_1, \tag{5}$$

where both A_1 and E_1 are including the phase shift and V_g denotes the group velocity. It is found that $T_a > 0$ for the antisymmetric mode of the modulational waves with $V_g > 0$, which means that the disturbances are amplified for the antisymmetric mode propagating to the positive direction of x or downstream.

NUMERICAL RESULTS

In order to confirm above results in the weakly nonlinear analysis and see the effects of larger nonlinearity, we make numerical analysis of eqs.(1) based on the steady solutions which is numerically obtained from eqs.(2). It is then found from Fig.2 for We=100 and Fr=$\sqrt{10}$ that the antisymmetric mode is amplified as the disturbances propagate downstream. It is also found from Fig.2(b) that the symmetric mode is locally induced at every half wave length in the antisymmetric mode and, as a result of this, it is expected that the sheet is disintegrated at the bottlenecks in this induced mode.



(a) $t = 20$ (b) $t = 50$

Fig.2 Propagations of the antisymmetric mode when We=100 and Fr=$\sqrt{10}$.

CONCLUSIONS

Resulting from the weakly nonlinear analysis and numerical analysis based on the sheet equations, we find that the antisymmetric mode of the disturbances propagating downstream is amplified. It is expected that the sheet is disintegrated due to the induced symmetric mode from the amplified antisymmetric mode.

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