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A GEOMETRICALLY NON-LINEAR FINITE SHELL ELEMENT WITH PIEZOELECTRIC LAYERS

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EXTENDED SUMMARY

In recent years the potential of integrating piezoelectric materials into smart structures has aroused the interest of many researchers. Theoretical models have been developed in order to fully understand the behaviour of smart structures with integrated piezoelectric materials. These models range from relatively simple linear beam theories to complex geometrically as well as physically non-linear models.
Many models are developed in order to investigate the mentioned smart structures. These models are based on either a beam or a plate/shell theory. Piefort [1] has pointed to the fact that the application of a beam theory to model thin smart structures will not produce satisfying results when collocated systems are considered. A plate or shell model is therefore to be preferred.
Considerably less work can be found based in the area of geometrically non-linear shell theories. This is quite surprising considering the fact that many authors refer to a particular benchmark problem also described in [1]. In this benchmark problem a cantilever piezoelectric bimorph beam acting as a sensor is deflected reasonably far away from the geometrically linear regime.
In this work a moderate rotation theory is deployed in the finite element formulation of a composite shell with integrated piezoelectric layers. The strain-displacement relations are valid for small strains but moderate rotations (see Schmidt and Reddy [2]).

A total Langrangian formulation is applied to define the internal virtual work. This requires not only the application of the 2nd Piola-Kirchhof stress and the Green-Lagrange strain tensor, but also the introduction of electrical field quantities defined in material curvilinear coordinates. The covariant elements of the electric field vector ${}_0E_i$ and the contravariant elements of the electric displacement vector ${}_0D^i$ in material curvilinear coordinates can be written as:

$${}_0E_i = -\frac{\partial \phi}{\partial \Theta^i} \qquad \text{and} \qquad {}_0D^i = J\,{}_tD^i, \tag{1}$$

where J denotes the determinant of the deformation gradient, Θ^i denotes the surface parameter in direction i, ϕ is the electric potential and the lower left indices 0 and t refer to quantities defined in material respectively spatial coordinates. It can be easily shown that both quantities ${}_0E_i$ and ${}_0D^i$ are energetically conjugated.
These new definitions together with the well-known linear direct and converse piezoelectric effect and the moderate rotation theory is then assembled into a finite shell element. An additional assumption is made in allowing only an electric field in transverse direction which is uniform between the positive and negative pole of an electrode pair.

Considering the benchmark problem discussed in [1] it can be shown that not only considerable differences exist between the analytical beam solution and the finite shell element approximation, but also between the geometrically linear and non-linear finite element results. These differences are even larger than the primarily mentioned ones. In the benchmark problem a cantilever piezoelectric bimorph beam is simulated $(100 \times 5 \times 1 \text{ mm})$, $E = 2.0$ GPa (Young's modulus), $d_{31} = 2.2 \cdot 10^{-11}$ Cb/N (piezoelectric constant) and $\delta_{33} = 1.062 \cdot 10^{-10}$ F/m (dielectric constant). In order to investigate the actuator and sensor properties of the beam it is either imposed with an electrical voltage or with a transverse tip force. In the latter case the beam is covered with five equal electrode pairs.
To make the differences between the results of beam analysis, the geometrically linear and non-linear finite element approximations more apparent, the boundary conditions have been changed from *clamped-free* to *clamped-hinged* in the actuator case and to *clamped-clamped* in the sensor case. The results are displayed in figure 1.

Actuator

In figure 1a the piezoelectric bimorph beam with the mentioned boundary conditions is imposed with 200 V. The finite element analysis is performed twice. One case is calculated with the normal boundary conditions as mentioned above, and the other case is calculated with adapted boundary conditions to invoke beam-like behaviour. Then both results are compared with the analytical result:

$$\frac{3d_{31}\phi}{4h^2L}\left(Lx^2 - x^3\right), \tag{2}$$

where h is the total thickness of the beam and L its length. From this comparison one can conclude that the difference between the geometrically non-linear finite shell element approximation and the analytical result is explained by the boundary conditions: Due to the clamping conditions in finite element analysis (MRT shell) high stress concentrations are induced in these areas. Once these effects are cancelled (MRT beam) the non-linear results agree well with the analytical results. It can then be concluded that the effects of the geometrical non-linearity is not profound in the actuator application.

Sensor

In figure 1b the bimorph beam is used as a sensor. Five equally large electrode pairs are attached to the beam and in the middle a transverse force acts downwards to obtain a mid-point deflection of 2 mm. Comparing the analytical results to the linear finite shell element approximation, the influence of the clamping conditions can be noticed in these areas.
Due to the induced membrane stresses, which are considered in the geometrically non-linear case, the required force to obtain a deflection of 2 mm (1.1843 N), is much greater than in the linear analysis (0.32133 N). For comparison purposes figure 1b additionally displays the results in case the force remains equal (deflection: 1.09 mm). Now one can conclude that the effect of the geometrically non-linearity is more profound than the clamping effect, in the sensor application.
Another phenomenon which appears in this case is the fact that the relative voltage distribution changes when the geometrically non-linear case is considered.

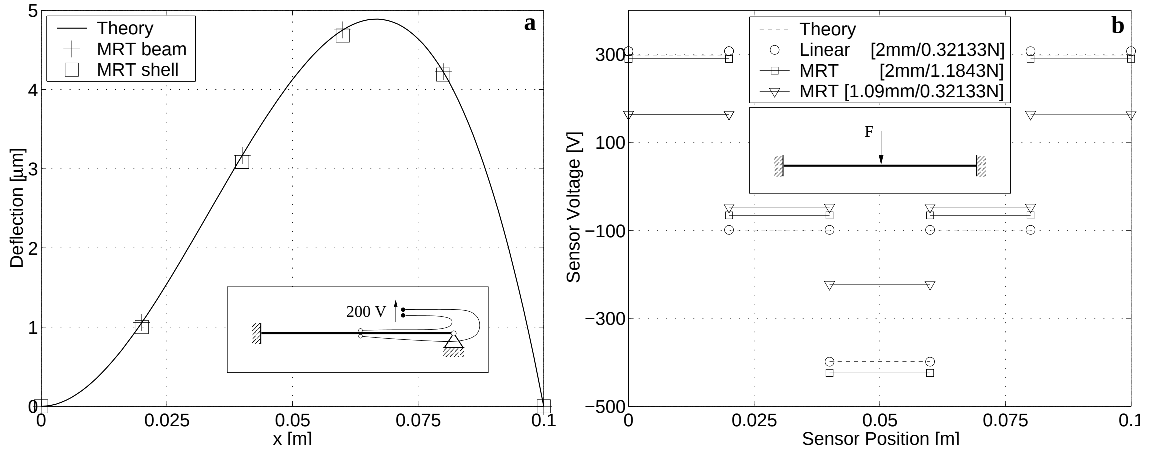

Figure 1. **a**: bimorph beam acting as an actuator with 200V. **b**: bimorph beam acting as sensor with a mid-point displacement of 2mm.

The effects shown in this benchmark problem, where a beam is clamped on both sides, are representative for many cases in which plates or shells as 2- or 3-dimensional objects are clamped along the sides. Not only the absolute sensor voltage, but also the relative distribution is influenced by the non-linearity. This could influence the effectivity of, for instance, modal damping algorithms used in linear dynamic systems.
Further examples, concerning beams/plates and shells with integrated piezoelectric layers, will be presented to validate the finite shell element formulation and to demonstrate the effect of geometrical non-linearity on actuator and sensor applications.

References

[1] Piefort, V.: Finite Element Modelling of Piezoelectric Active Structures. *PhD Thesis*, Université Libre de Bruxelles, 2001.
[2] Schmidt R., Reddy J.: A Refined Small Strain and Moderate Rotation Theory of Elastic Anistropic Shells. *J. Appl Mech* **55**:611-617, 1988.