A LAGRANGIAN APPROACH TO WAVE-INDUCED OCEANIC MASS TRANSPORT

Jan Erik Weber
Department of Geosciences, University of Oslo
P. O. Box 1022, Blindern, N-0315 Oslo, Norway

Summary The mean mass transport induced by surface gravity waves is investigated theoretically for a deep, rotating ocean with a constant eddy viscosity. The waves are periodic in space, and have amplitudes that grow or decay slowly in time. The analysis is based on a Lagrangian description of motion, and the results are valid to second order in the wave steepness. Various oceanic applications are discussed.

INTRODUCTION

It is well-known that surface water waves carry mean momentum [1]. For monochromatic waves in a viscous non-rotating fluid, the pioneering paper is [2]. For a direct Lagrangian approach to wave drift in a rotating ocean, earlier treatments of this problem are found in papers such as [3], [4], and [5]. Also the generalized Lagrangian-mean formulation in [6] can be applied to this problem. When the wind is blowing over the sea, it exerts a mean stress along the sea surface. This stress may balance the Coriolis force to yield an Ekman transport in the oceanic surface layer. The Ekman transport, and the vertical motion associated with the horizontal convergence or divergence of this transport, is crucial for the understanding of the general ocean circulation. However, the wind will also generate surface waves, and with the present-day modeling tools, e.g. [7], the surface wind wave spectrum is well resolved. One of the remaining problems, as far as the ocean circulation is concerned, is to determine in a realistic way the mass flux induced by the wind waves. In [8] this problem has been investigated for time-periodic waves with spatially-varying amplitudes. We here study the similar problem for surface waves with amplitudes that decay or grow slowly in time.

MATHEMATICAL FORMULATION

Wave momentum is related to particle motion in waves, and is most conveniently discussed by using a Lagrangian description of motion. We consider plane waves that propagate along the x-axis in a Cartesian coordinate system, where the x- and y-axes are situated at the undisturbed horizontal sea surface. The z-axis is vertical, and directed upwards. Our system rotates with constant angular velocity $f/2$ about the vertical axis, where f is the Coriolis parameter. The ocean is incompressible, and we apply an eddy assumption for the turbulent Reynolds stresses. We here consider deep-water surface gravity waves. A fluid particle is associated with its initial coordinates (a, b, c). The particle position at later times (X, Y, Z) and the pressure P will then be functions of a, b, c, and time t. Velocity components and accelerations are given by (X_t, Y_t, Z_t), and (X_{tt}, Y_{tt}, Z_{tt}), respectively, where subscripts denote partial differentiation. In a Lagrangian description the free material surface is given by $c = 0$. For plane waves along the x-axis, the deviations (x, y, z, p) from the initial state will not depend on b. We then may write

$$X = a + x(a,c,t),\ Y = b + y(a,c,t),\ Z = c + z(a,c,t),\ P = -\rho g c + p(a,c,t)\,,$$

where ρ denotes the constant density and g is the acceleration due to gravity. For convenience, we introduce a complex horizontal velocity $W = x_t + i y_t$, where i is the imaginary unit. With the present notation, the momentum equations become to second order in wave steepness

$$W_t + ifW - \nu\nabla_L^2 W = -\frac{1}{\rho}(p+\rho g z)_a - \frac{1}{\rho}J(p,z) + \nu\{J(W_a,z) + J(x,W_c) + J(W,z)_a + J(x,W)_c\},$$

$$z_{tt} - \nu\nabla_L^2 z_t = -\frac{1}{\rho}(p+\rho g z)_c - \frac{1}{\rho}J(x,p) - gJ(x,z) + \nu\{J(z_{ta},z) + J(x,z_{tc}) + J(z_t,z)_a + J(x,z_t)_c\}.$$

Here ν is the constant turbulent eddy viscosity, and $\nabla_L^2 \equiv \partial^2/\partial a^2 + \partial^2/\partial c^2$ is the Laplacian operator in Lagrangian coordinates. Furthermore, $J(A,B) \equiv A_a B_c - A_c B_a$ is the Jacobian. Finally, the conservation of mass (here volume) leads to

$$x_a + z_c = -J(x,z)\,.$$

We consider wave motion that is periodic in space, and average our variables over one wavelength. When we generalize, and move from considering a single wave component to a fully developed wind sea, this process must be understood as an ensemble mean. Denoting the averaging process by an over-bar, and integrating the averaged horizontal momentum equations from a depth *H*, where the motion and the stresses vanish, to the surface, we obtain

$$\frac{d}{dt}\int_{-H}^{0}\rho\overline{W}dc + if\int_{-H}^{0}\rho\overline{W}dc = \rho\nu\overline{W}_c(c=0) - F_w \,. \qquad \text{(I)}$$

Here $\overline{W}_c(c=0)$ depends on the external stresses and the physical conditions at the sea surface [9]. When the viscous effect of the air is neglected, we find that $\overline{W}_c(c=0)=0$, e.g. [2]. F_w in (I) contains all the nonlinear terms in the horizontal momentum equation. For wind waves with frequencies much higher than *f*, we may write

$$F_w = \int_{-H}^{0}\{(\overline{p}+\rho g\overline{z})_a + \overline{J(p,z)} - \rho\nu(\overline{J(x_{ta},z)} + \overline{J(x,x_{tc})} + \overline{J(x_t,z)}_a + \overline{J(x,x_t)}_c)\}dc.$$

MAIN RESULT AND DISCUSSION

Inserting for the linear, real, deep-water wave field (denoted by the symbol ~), with frequency ω, growth rate β, wave number k and initial amplitude ζ_0 [9] (neglecting the effect of the fluctuating tangential wind stress) into the right-hand side of (I), we find for the wind- and wave-induced complex Lagrangian mean mass transport

$$\frac{d}{dt}\int_{-H}^{0}\rho\overline{W}dc + if\int_{-H}^{0}\rho\overline{W}dc = \tau_E + \tau_D \,. \qquad \text{(II)}$$

Here τ_E denotes the mean tangential (Ekman) wind stress component, and τ_D is the form drag given by

$$\tau_D \equiv -\overline{\tilde{\sigma}\tilde{z}_a}, \quad c=0\,,$$

where $\tilde{\sigma}$ is the real part of the fluctuating wind-stress normal to the sea surface. In the present formulation we find

$$\tau_D = \frac{d}{dt}M + 2\rho\nu k^2\omega\zeta_0^2\exp(2\beta t)\,. \qquad \text{(III)}$$

Here $M = \frac{1}{2}\rho\omega\zeta_0^2\exp(2\beta t)$ is the total horizontal wave momentum. (It may be obtained by integrating the Stokes drift in the vertical, [2]). We note that in the absence of wind, we have $\tau_D = 0$, or $\beta = -2\nu k^2$ for temporally decaying waves, and hence the total wave-induced mass transport is zero when averaged over the inertial period [9]. Various aspects of (II) related to growing and decaying (breaking) waves are discussed. For example, when generalized to a statistically steady sea state $(\beta = 0)$, empirical models for τ_D in (III) can be used to relate the turbulent bulk eddy viscosity in the water to the properties of the wave spectrum and the wind speed.

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