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NONLINEAR AFFINE EXTENSION OF THE THREE-PHASE SPHERE MODEL

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Summary To model the nonlinear behaviour of particulate composites, the classical procedure of transforming, at a given strain, a nonlinear problem into a linear one has been used. The chosen linearisation method is the recently proposed affine formulation [1, 2] which has been coupled to the three phase self-consistent estimate [3] of the linearised overall properties, in the case of uniaxial loading. This modelling has been applied to two-phase composites with nonlinear elastic behaviours described by power-law stress-strain relationships, but the same developments are still valid for viscous materials.

AFFINE PROCEDURE

According to the affine procedure, the nonlinear behaviour of each phase r ($\sigma = f_r(\varepsilon)$) is approximated, at a given reference strain ε_r^0 which is set equal to the average local strain in the phase, by a thermoelastic linear behaviour defined by the tangent moduli L_r and the pre-stress τ_r^0, both being taken constant per phase:

$$\langle\sigma\rangle_r = L_r : \langle\varepsilon\rangle_r + \tau_r^0 \quad , \quad L_r = \frac{df_r}{d\varepsilon}(\varepsilon_r^0) \quad , \quad \tau_r^0 = f_r(\varepsilon_r^0) - L_r : \varepsilon_r^0 \tag{1}$$

Such a linearisation procedure leads to the definition of a Linear-thermoelastic Comparison heterogeneous Medium (LCM) whose effective moduli $\tilde{L}$ have to be estimated by a linear homogenisation scheme. An appropriate choice for a matrix-inclusion morphology is the three-phase self-consistent model, which needs recourse to a numerical resolution because of the anisotropic tangent behaviour. After numerical computation of $\tilde{L}$, the average local strains $\langle\varepsilon\rangle_r$ are given, for two-phase thermoelastic composites, by Levin's relation [4]:

$$\langle\varepsilon\rangle_r = A_r : E - (A_r - I) : (L_1 - L_2)^{-1} : (\tau_1^0 - \tau_2^0) \tag{2}$$

where E is the prescribed macroscopic strain and A_r the averaged linear localisation tensors for each phase r, i. e.:

$$A_1 = \frac{1}{c_1}(\tilde{L} - L_2) : (L_1 - L_2)^{-1} \quad , \quad A_2 = \frac{1}{c_2}(I - c_1 A_1) \tag{3}$$

where c_1 and c_2 are the concentrations of phase 1 and 2 and I is the fourth-order unit tensor.
The two steps of linearisation and estimation of the effective properties of the LCM are repeated until, as prescribed by the affine method, the reference strains ε_r^0 are equal to the average strains in the corresponding linearised phase $\varepsilon_r^0 = \langle\varepsilon\rangle_r$. After convergence, the overall stress of the composite is given by:

$$\Sigma = \tilde{L} : E + \Sigma^0 \quad , \quad \tilde{L} = \langle L : A\rangle \quad , \quad \Sigma^0 = \langle {}^tA : \tau^0\rangle \tag{4}$$

NUMERICAL 3-PHASE SELF-CONSISTENT ESTIMATE FOR TRANSVERSELY ISOTROPIC PHASES

Due to the anisotropic tangent behaviour, the linear three-phase self-consistent estimate (or generalised self consistent estimate) for the effective properties of the LCM has to be computed numerically. The 3-phase sphere model, initially proposed by Christensen and Lo [3], is devoted to composites constituted by a continuous matrix in which inclusions are isotropically distributed. It refers to Hashin's assemblage of homothetic composite spherical inclusions mapping the whole representative volume element: the inclusions are constituted of a sphere of filler phase, coated by a concentric shell of matrix phase with the same volume fractions as the whole composite. The resolution of the three-phase model requires to determine the linear mechanical response of the composite inclusion embedded in an infinite reference medium with elastic moduli L^0 and subjected to the homogeneous displacement field $E^0 x$ at infinity. The overall elastic tensor $\tilde{L}_{HS}(L^0)$ verifying $\langle\sigma\rangle = \tilde{L}_{HS}(L^0)\langle\varepsilon\rangle$, is the so-called generalised Hashin-Shtrikman estimate, which, depending on the choice of the reference medium, corresponds to bounds or estimates [5]. The self-consistent estimate is generated when the reference medium has the properties of the homogenised equivalent medium itself, i. e., $L^0 = \tilde{L}_{HS}(L^0)$.
This problem has been addressed by the Finite Elements method by means of the FEM code Cast3M in the case of uniaxial loadings of unit vector n, leading to transversely isotropic tangent stiffness tensors, noted in the form [6]:

$$L_r = \alpha E_L + \beta J_T + \gamma F + \gamma'^T F + \delta K_T + \delta' K_L \tag{5}$$

where $E_L = n \otimes n \otimes n \otimes n$ is the projector on uniaxial tensors, $J_T = \frac{1}{2} i_T \otimes i_T$ is the projector on the identity in the transverse plane ($i_T = i - n \otimes n$, with i the second-order unit tensor), $K_L = K - K_T - K_E$ (with $K_E = \frac{1}{6}(2n \otimes n - i_T) \otimes (2n \otimes n - i_T)$) is the projector on longitudinal shears, $K_T = I_T - J_T$ (I_T being the fourth-order identity tensor in the transverse plane) is the projector on transverse shears and $F = \frac{1}{\sqrt{2}} i_T \otimes n \otimes n$ is related to Poisson's effect.

Five independent constants have to be found for each phase (eq. 5): α, β, γ and γ', characterizing the response to axisymmetric loadings, and δ and δ', defining the shear behaviour. These components are obtained by the computation of the average stress and strain tensors in the composite inclusion for four applied strain conditions (uniaxial load, plane compression, longitudinal shear and transverse shear). Using axisymmetric and Fourier modes, these calculations have been conducted on two-dimensional axisymmetric cells (fig. 1 (a)). To get the self-consistent estimate we used an iterative procedure: starting from an initial reference medium L^0, we compute an estimate $\tilde{L}_{HS}$ which is then used as a new reference medium; this procedure is repeated until the self-consistent condition is achieved within a given convergence criterium.

APPLICATIONS

This treatment has been applied to particulate composite made of isotropic power-law phases with the local constitutive equations: $\varepsilon^{eq} = \sigma^{eq}/3\mu + \varepsilon_0(\sigma^{eq}/\sigma_0)^n$ where μ is the shear modulus, ε^{eq} and σ^{eq} are the equivalent Von Mises strain and stress, ε_0 is a reference strain and σ_0 characterises the stiffness of the phase. Fig. 1 (b) shows the effective behaviour of a composite with 30% inclusions obtained for a contrast of 5 between phases: $\sigma_0^{(2)} = 5\sigma_0^{(1)}$ and, for each phase, $n = 5$, $\varepsilon_0 = 1$, $\sigma_0/E = 10^{-3}$, $\nu = 0.3$. One can observe that the affine three-phase self-consistent estimate does not violate the upper bound for such nonlinear composite with a particulate microstructure obtained by the combination of the variational procedure for nonlinear behaviour of Ponte Castañeda [7] and a linear upper bound due to Hervé, Stolz and Zaoui [8]. This new model predicts a stiffer behaviour than the modified secant three-phase self-consistent scheme, which is however limited to nonlinear elasticity and viscoplasticity, which is not the case of the affine procedure. Moreover, the comparison with preliminary experimental data obtained for a silica-filled polymer has shown a great agreement [9].

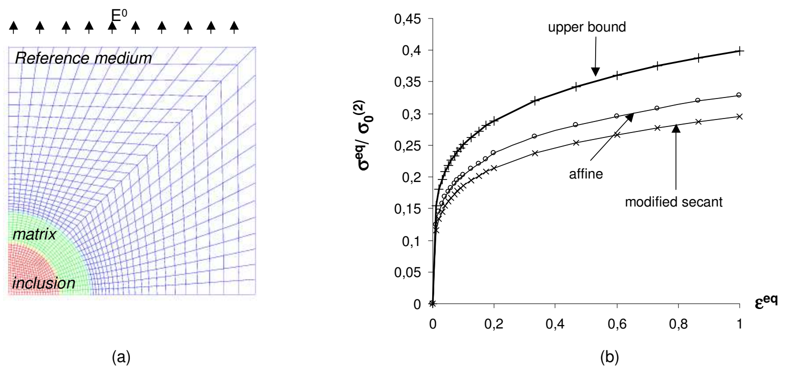

Figure 1. (a) Example of 2D mesh used. (b) Effective behaviour of a two-phase, power-law type, particulate composite with 30% inclusions and a contrast of 5.

CONCLUSION

The modelling of nonlinear composites with a matrix-inclusion microstructure has been achieved by the combination of the affine procedure with the three-phase sphere self-consistent estimate, in the framework of nonlinear elasticity and for uniaxial loads. Such a linearisation procedure presents also the advantage, with respect to other linearisation schemes, to be able to deal with hereditary behaviours, such as elastoviscoplasticity. Further work is under program to extend the present procedure to such behaviours.

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