XXI ICTAM, 15–21 August 2004, Warsaw, Poland

INHOMOGENEOUS CIRCULARLY POLARIZED WAVES IN ORTHORHOMBIC CRYSTALS

Philippe Boulanger*, Michael Hayes**
*Université Libre de Bruxelles, Dpt. de Mathématique, Campus Plaine CP218/1, 1050 Bruxelles, Belgium
**University College Dublin, Department of Mechanical Engineering, Belfield, Dublin 4, Ireland

Summary For elastic homogeneous plane waves in crystals, Fedorov § Fedorov introduced a decomposition of the acoustical tensor, valid for all orthorhombic, tetragonal, hexagonal and cubic crystals. Here, in considering the propagation of inhomogeneous waves in orthorhombic crystals, we generalize the Fedorov § Fedorov concept of "pseudo-transverse" and "pseudo-longitudinal" waves to elliptically polarized inhomogeneous waves, and determine the corresponding possibilities for circularly polarized waves.

BASIC EQUATIONS

Within the context of linearized elasticity theory, the equations of motion governing the displacement u_i of an anisotropic elastic body of material density ρ are

$$d_{ijkl}\partial^2 u_k/\partial x_j \partial x_l = \rho \partial^2 u_i/\partial t^2, \tag{1}$$

where d_{ijkl} are the elastic stiffnesses, assumed constant, with the symmetries $d_{ijkl} = d_{jikl} = d_{ijlk} = d_{klij}$. For orthorhombic crystals, using Voigt's notation, the only non-zero elastic stiffnesses are $d_{11}, d_{22}, d_{33}, d_{44}, d_{55}, d_{66}, d_{12}, d_{13}, d_{23}$. The propagation of time-harmonic inhomogeneous plane waves with complex slowness $\mathbf{S} = N\mathbf{C}$ and complex amplitude $\mathbf{A}$ is governed by the propagation condition

$$\mathbf{Q}(\mathbf{C})\mathbf{A} = w\mathbf{A} \ , \quad Q_{ik}(\mathbf{C}) = d_{ijkl}C_j C_l \ , \quad \text{with } w = \rho N^{-2}. \tag{2}$$

The tensor $\mathbf{Q}(\mathbf{C})$ is called the "complex acoustical tensor". For each chosen bivector $\mathbf{C}$, the eigenvalue problem $(2)_1$ for w and $\mathbf{A}$ has to be solved. This procedure is called the "directional-ellipse method", or "DE-method"[4].

FEDOROV § FEDOROV DECOMPOSITION

For crystals of orthorhombic, tetragonal, hexagonal or cubic symmetry, Fedorov & Fedorov[1] introduced a decomposition of the acoustical tensor. Generalizing this decomposition to the case of inhomogeneous waves, the complex acoustical tensor $\mathbf{Q}(\mathbf{C})$ may be written as

$$\mathbf{Q} = \mathbf{D} + \mathbf{N} \otimes \mathbf{N} \ , \tag{3}$$

where $\mathbf{D}$ is a complex diagonal tensor and $\mathbf{N}$ a bivector, given by

$$\mathbf{D} = \text{diag}[D_1, \ D_2 \ , D_3] \ , \qquad \mathbf{N} = (\beta_1 C_1, \beta_2 C_2, \beta_3 C_3) \ . \tag{4}$$

For orthorhombic crystals, we have

$$\begin{gathered} D_1 = (d_{11} - \beta_1^2)C_1^2 + d_{66}C_2^2 + d_{55}C_3^2 \ , \quad D_2 = d_{66}C_1^2 + (d_{22} - \beta_2^2)C_2^2 + d_{44}C_3^2 \ , \\ D_3 = d_{55}C_1^2 + d_{44}C_2^2 + (d_{33} - \beta_3^2)C_3^2 \ , \end{gathered} \tag{5}$$

and

$$\beta_1^2 = \frac{(d_{12} + d_{66})(d_{13} + d_{55})}{(d_{23} + d_{44})} \ , \ \beta_2^2 = \frac{(d_{23} + d_{44})(d_{12} + d_{66})}{(d_{13} + d_{55})} \ , \ \beta_3^2 = \frac{(d_{13} + d_{55})(d_{23} + d_{44})}{(d_{12} + d_{66})} \ . \tag{6}$$

With the Fedorov & Fedorov decomposition (3), the secular equation, $\det\{\mathbf{Q}(\mathbf{C}) - w\mathbf{1}\} = 0$, reads

$$(w - D_1)(w - D_2)(w - D_3) - N_1^2(w - D_2)(w - D_3) - N_2^2(w - D_3)(w - D_1) - N_3^2(w - D_1)(w - D_2) = 0. \tag{7}$$

In considering the decomposition (3) of the acoustical tensor for homogeneous waves, Fedorov & Fedorov[1] introduced "pseudo-longitudinal" and "pseudo-transverse" waves. "Pseudo-longitudinal" waves are those waves whose amplitude $\mathbf{A}$ is along $\mathbf{N}$, whilst "pseudo-transverse" waves are those waves whose amplitude $\mathbf{A}$ is orthogonal to $\mathbf{N}$. Here, we consider "pseudo-longitudinal" and "pseudo-transverse" *inhomogeneous* waves, and analyze the possibility of circularly polarized waves. Circularly polarized waves are possible if and only if the secular equation (7) has a double root.

PSEUDO-LONGITUDINAL WAVES

Pseudo-longitudinal waves ($\mathbf{N} \times \mathbf{A} = \mathbf{0}$) are only possible when $\mathbf{N}$ satisfies the condition[1] $\mathbf{N} \times \mathbf{QN} = \mathbf{0}$. This is a condition on the bivector $\mathbf{C}$. From (3), it is easily seen that this reduces to $\mathbf{N} \times \mathbf{DN} = \mathbf{0}$, which means that $\mathbf{DN}$ has to be parallel to $\mathbf{N}$, and thus

$$(D_1 N_1)/N_1 = (D_2 N_2)/N_2 = (D_3 N_3)/N_3 \ . \tag{8}$$

Because one or two of the components of $\mathbf{N}$ may be zero, three possibilities have to be considered. Here we consider only the general case general when $N_1N_2N_3 \neq 0$ for which circularly polarized waves are possible. In this case, the conditions (8) reduces to $D_1 = D_2 = D_3$. These are two equations for the ratios C_2/C_1 , C_3/C_1. The real solutions yield acoustic axes of the crystal[3], along which circularly polarized homogeneous waves may propagate. However, depending on the elastic stiffnesses, complex solutions are also possible. These yield bivectors $\mathbf{C}$ for which the secular equation has a double root. The corresponding solutions are given by

$$w_1 = D_1 + \mathbf{N}\cdot\mathbf{N}\ , \quad \mathbf{A}_1 = \mathbf{N}\ ; \qquad w_2 = w_3 = D_1\ , \quad \mathbf{N}\cdot\mathbf{A} = 0\ . \tag{9}$$

The solution $(w_1,\ \mathbf{A}_1)$ represents a pseudo-longitudinal wave, whilst the solution $(w_2,\ \mathbf{A})$ represents a pseudo-transverse wave with an arbitrary amplitude $\mathbf{A}$ orthogonal to $\mathbf{N}$. In particular, for this wave, $\mathbf{A}$ may be chosen to be isotropic, $\mathbf{A}\cdot\mathbf{A} = 0$ and then, the wave is circularly polarized.

PSEUDO-TRANSVERSE WAVES

Pseudo-transverse waves ($\mathbf{N}\cdot\mathbf{A} = 0$) are only possible when $\mathbf{N}$ satisfies the condition[1] $(\mathbf{N}\times\mathbf{QN})\cdot\mathbf{Q}^2\mathbf{N} = 0$, which means that $\mathbf{N}$, $\mathbf{QN}$, $\mathbf{Q}^2\mathbf{N}$ are linearly dependent. From (3), it is easily seen that this reduces to $(\mathbf{N}\times\mathbf{DN})\cdot\mathbf{D}^2\mathbf{N} = 0$, which may be written explicitly as[1]

$$N_1N_2N_3(D_1 - D_2)(D_2 - D_3)(D_3 - D_1) = 0. \tag{10}$$

Two possibilities have to be considered, namely when one of the N_i is zero, or when two of the D_i are equal. As an example, we here present results for the case $D_1 = D_2$. In this case the secular equation (7) is factored so that we have the solution

$$w_1 = D_1 \quad , \qquad \mathbf{A}_1 = (-N_2, N_1, 0)\ , \tag{11}$$

whilst the two other roots $w = w_2, w_3$ of (7) are the roots of the quadratic

$$(w - D_3)(w - D_1 - \mathbf{N}\cdot\mathbf{N}) + N_3^2(D_1 - D_3) = 0\ . \tag{12}$$

Clearly, (11) represents a pseudo-transverse wave ($\mathbf{N}\cdot\mathbf{A} = 0$). The two roots of (12) yield two other wave solutions. Circularly polarized waves are possible either (a) when w_1 given by (11) is also a root of of the quadratic (12), or (b) when the quadratic (12) has a double root.

Case (a) The root $w_1 = D_1$ is also a root of (12) when

$$(D_1 - D_3)(N_1^2 + N_2^2) = 0\ . \tag{13}$$

We note that for inhomogeneous waves, (13) may be satisfied by choosing $N_2 = \pm iN_1$. In this case, there is a simple infinity of isotropic eigenbivectors corresponding to the double root w_1 of the secular equation (7). We have

$$w_1 = D_1 \quad , \qquad \mathbf{A}_1 = (\pm i, -1, 0)\ . \tag{14}$$

This solution represents an inhomogeneous plane wave which is circularly polarized in the x_1x_2-plane.

Case (b) The quadratic (12) has a double root when

$$(D_1 - D_3 + N_1^2 + N_2^2 - N_3^2)^2 + 4(N_1^2 + N_2^2)N_3^2 = 0\ . \tag{15}$$

For inhomogeneous waves, this condition may be satisfied by requiring that

$$D_1 - D_3 + N_1^2 + N_2^2 - N_3^2 = \pm 2i(N_1^2 + N_2^2)^{1/2}N_3\ . \tag{16}$$

We here assume $N_1N_2N_3 \neq 0$. Then, when (16) is satisfied, the secular equation (7) has a double root w, and corresponding to this double root there is a simple infinity of isotropic eigenbivectors $\mathbf{A}$. These are given by

$$w = \frac{1}{2}(D_1 + D_3 + \mathbf{N}\cdot\mathbf{N}) \quad , \qquad \mathbf{A} = (iN_1, iN_2, \pm(N_1^2 + N_2^2)^{1/2})\ . \tag{17}$$

This solution represents a circularly polarized inhomogeneous plane wave. Using (5) and (6), the results may be explicited in terms of the elastic stiffnesses.

References

[1] Fedorov, F. I. & Fedorov, A. F.: On the structure of the Green-Christoffel tensor. *J. Phys. A: Math. Gen.* **24**:71–78, 1991.
[2] Boulanger, Ph. & Hayes, M.: Bounds on elastic wave-speeds in crystals: Theory and applications. *Proc. Roy. Soc. A* **454**:2298-2322, 1998.
[3] Boulanger, Ph. & Hayes, M.: Acoustic axes for elastic waves in crystals: Theory and applications. *Proc. Roy. Soc. A* **454**:2323-2346, 1998.
[4] Boulanger, Ph. & Hayes, M.: Bivectors and waves in mechanics and optics. Chapman & Hall, London 1993.