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EXTENDED POLAR DECOMPOSITIONS FOR FINITE PLANE STRAIN

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Summary The concept of unsheared triads of material line elements in a body was introduced by Boulanger & Hayes who showed that there is a link between these triads and new decompositions of the deformation gradient, called "extended polar decompositions", generalizing the classical polar decomposition. In the present paper attention is confined to finite plane strain so that the deformation gradient is essentially two-dimensional. The whole range of corresponding extended polar decompositions is presented.

BASIC EQUATIONS

Let a body of material be subjected to the deformation

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \ , \quad x_i = x_i(X_A) \ , \qquad (i, A = 1, 2, 3), \tag{1}$$

in which the particle initially at $\mathbf{X}$ is displaced to $\mathbf{x}$. All quantities are referred to a fixed rectangular Cartesian coordinate system. A material line element $d\mathbf{X}$ at $\mathbf{X}$ is deformed into the element $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, where $\mathbf{F}$, the deformation gradient at $\mathbf{X}$ is given by $F_{iA} = \partial x_i/\partial X_A$. The right Cauchy-Green strain tensor $\mathbf{C}$, and the left $\mathbf{B}$, are given by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \ , \quad C_{AB} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B} \ , \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T \ , \quad B_{ij} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_j}{\partial X_A} \ . \tag{2}$$

If $\mathbf{N}$ is a unit vector along a material line element of infinitesimal length L before deformation, at $\mathbf{X}$, then, after the deformation the length of the element at $\mathbf{x}$ is $\lambda(\mathbf{N})L$, where $\lambda(\mathbf{N})$, called the "stretch along $\mathbf{N}$", is given by

$$\lambda_{(\mathbf{N})} = |\mathbf{F}\mathbf{N}| = (\mathbf{N} \cdot \mathbf{C}\mathbf{N})^{1/2} = (C_{AB}N_AN_B)^{1/2} \ . \tag{3}$$

Similarly, if $\hat{\mathbf{n}}$ is a unit vector along a material line element of infinitesimal length l after deformation, at $\mathbf{x}$, then, before the deformation the length of the element at $\mathbf{X}$ is $\lambda(\mathbf{n})l$, where

$$\lambda_{(\mathbf{n})} = (\hat{\mathbf{n}} \cdot \mathbf{B}^{-1}\hat{\mathbf{n}})^{1/2} = (B^{-1}_{ij}\hat{n}_i\hat{n}_j)^{1/2} \ . \tag{4}$$

If $\mathbf{n} = \mathbf{F}\mathbf{N}$, so that $\mathbf{n}$, along $\hat{\mathbf{n}}$, is the element into which $\mathbf{N}$ is deformed, then $\lambda_{(\mathbf{n})} = 1/\lambda_{(\mathbf{N})}$.

UNSHEARED TRIADS

Unsheared triads consist of the three material line elements at $\mathbf{X}$ along the unit vectors $\mathbf{M}, \mathbf{N}, \mathbf{P}$, such that the three pairs of material line elements along $(\mathbf{M}, \mathbf{N}), (\mathbf{N}, \mathbf{P})$ and $(\mathbf{P}, \mathbf{M})$ are unsheared, i.e. suffer no change in angle. Recently, it has been shown[2] that, in any deformation, there is an infinity of unsheared triads at $\mathbf{X}$.
The conditions for $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{P}$ to form an unsheared triad are (e. g. Truesdell & Toupin[3])

$$\mathbf{M}\cdot\mathbf{C}\mathbf{N} = \lambda_{(\mathbf{N})}\lambda_{(\mathbf{M})}\mathbf{M}\cdot\mathbf{N} \ , \ \ \mathbf{N}\cdot\mathbf{C}\mathbf{P} = \lambda_{(\mathbf{N})}\lambda_{(\mathbf{P})}\mathbf{N}\cdot\mathbf{P} \ , \ \ \mathbf{P}\cdot\mathbf{C}\mathbf{M} = \lambda_{(\mathbf{P})}\lambda_{(\mathbf{M})}\mathbf{P}\cdot\mathbf{M} \ . \tag{5}$$

Assuming that an unsheared pair $(\mathbf{M}, \mathbf{N})$ is known, and considering only non coplanar triads, a formula for $\mathbf{P}$ forming with $\mathbf{M}$ and $\mathbf{N}$ an unsheared triad is[2]

$$p\mathbf{P} = \{\mathbf{C}\mathbf{M} - \lambda^{-1}_{(\mathbf{N})}(\det\mathbf{C})^{1/2}\mathbf{M}\} \times \{\mathbf{C}\mathbf{N} - \lambda^{-1}_{(\mathbf{M})}(\det\mathbf{C})^{1/2}\mathbf{N}\} \ , \tag{6}$$

where p is a scalar factor such that $\mathbf{P} \cdot \mathbf{P} = 1$.
Thus, in general, for any chosen unsheared pair $(\mathbf{M}, \mathbf{N})$, there is a unique unit vector $\mathbf{P}$ (by "unique", we here mean unique up to a $\pm$ sign) such that $(\mathbf{M}, \mathbf{N}, \mathbf{P})$ is an unsheared triad. In other words, if an unsheared pair of material line elements is given at a point $\mathbf{X}$, then, in general, a unique third material line element at $\mathbf{X}$ may be found such that the three material line elements form an unsheared triad.
However, special cases may occur because for special choices of the unsheared pair $(\mathbf{M}, \mathbf{N})$, (6) yields $p\mathbf{P} = \mathbf{0}$, or $p\mathbf{P}$ along $\mathbf{M}$ or $\mathbf{N}$ (see [2] for details).

EXTENDED POLAR DECOMPOSITIONS

Let $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P}$ be unit vectors along the edges of a (non coplanar) unsheared triad of material line elements, and let $\mathbf{m} = \mathbf{F}\mathbf{M}$, $\mathbf{n} = \mathbf{F}\mathbf{N}$, $\mathbf{p} = \mathbf{F}\mathbf{P}$. It has been shown [2] that $\mathbf{F}$, the deformation gradient, may be written

$$\mathbf{F} = \mathbf{Q}\mathbf{G} = \mathbf{H}\mathbf{Q} \ , \tag{7}$$

where $\mathbf{Q}$ is a proper orthogonal tensor($\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$, $\det \mathbf{Q} = 1$), and $\mathbf{G}$, $\mathbf{H}$ are given by

$$\mathbf{G} = \lambda_{(\mathbf{M})} \mathbf{M} \otimes \mathbf{M}^* + \lambda_{(\mathbf{N})} \mathbf{N} \otimes \mathbf{N}^* + \lambda_{(\mathbf{P})} \mathbf{P} \otimes \mathbf{P}^* \ , \tag{8}$$

$$\mathbf{H} = \lambda^{-1}_{(\mathbf{m})} \mathbf{m} \otimes \mathbf{m}^* + \lambda^{-1}_{(\mathbf{n})} \mathbf{n} \otimes \mathbf{n}^* + \lambda^{-1}_{(\mathbf{p})} \mathbf{p} \otimes \mathbf{p}^* \ . \tag{9}$$

Here, $(\mathbf{M}^*, \mathbf{N}^*, \mathbf{P}^*)$ is the triad reciprocal to $(\mathbf{M}, \mathbf{N}, \mathbf{P})$, and $(\mathbf{m}^*, \mathbf{n}^*, \mathbf{p}^*)$ is the triad reciprocal to $(\mathbf{m}, \mathbf{n}, \mathbf{p})$. The decompositions (7) have echoes of the classical polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, in which $\mathbf{R}$ is proper orthogonal and $\mathbf{U}$, $\mathbf{V}$ are positive definite symmetric. They are called "extended polar decompositions"[2]. Because there is an infinity of unsheared triads, there is an infinity of such "extended polar decompositions". Also we note that $\mathbf{C} = \mathbf{G}^T\mathbf{G}$ and $\mathbf{B} = \mathbf{H}\mathbf{H}^T$.

PLANE STRAIN

Here we consider plane deformations

$$x_\alpha = x_\alpha(X_\Lambda) \ , \quad x_3 = X_3 \ , \qquad (\alpha, \Lambda = 1, 2) \ . \tag{10}$$

Let $\mathbf{F}'$ denote the two by two deformation gradient $F'_{\alpha\Lambda} = \partial x_\alpha / \partial X_\Lambda$, and $\mathbf{C}'$, $\mathbf{B}'$ be the corresponding two by two right and left Cauchy-Green strain tensors.

We now restrict our attention to unsheared triads $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P}$ consisting of material line elements along two vectors $\mathbf{M}$, $\mathbf{N}$ in the X_1X_2–plane, and the vector $\mathbf{P} = \mathbf{K} = (0, 0, 1)$ along the X_3–axis. Thus, $(\mathbf{M}, \mathbf{N})$ may be any unsheared pair in the X_1X_2–plane. There is an infinity of such pairs : if $\mathbf{N}$ is given in the plane, there is, in general, a unique companion $\mathbf{M}$ forming with $\mathbf{N}$ an unsheared pair[1]. Here, we may obtain the companion $\mathbf{M}$ of a given $\mathbf{N}$ by using (6), because $\mathbf{M}$ is the third edge af an unsheared triad whose first two edges are $\mathbf{N}$ and $\mathbf{P} = \mathbf{K}$. If Ψ denotes the arbitrary angle that $\mathbf{N}$ makes with the X_1–axis, it follows that (omitting a scalar factor in $\mathbf{M}$)

$$\mathbf{N} = (\cos\Psi, \sin\Psi, 0) \ , \ \mathbf{M} = ([\det \mathbf{C}^{1/2} - C_{22}] \sin\Psi - C_{12}\cos\Psi, \ C_{12}\sin\Psi - [\det \mathbf{C}^{1/2} - C_{11}]\sin\Psi, \ 0) \ . \tag{11}$$

Then, using (8) with $\mathbf{P} = \mathbf{K} = \mathbf{P}^*$, and retaining only the components in the X_1X_2–plane, we have

$$\mathbf{G}' = \lambda_{(\mathbf{M})} \mathbf{M} \otimes \mathbf{M}^* + \lambda_{(\mathbf{N})} \mathbf{N} \otimes \mathbf{N}^* \ . \tag{12}$$

We give explicit details in terms of the angle Ψ.

Similarly, using the angle ϕ that $\mathbf{n} = \mathbf{F}\mathbf{N}$ makes with the x_1–axis, and the components of $\mathbf{B}^{-1}$, an expression may be obtained for $\mathbf{H}'$:

$$\mathbf{H}' = \lambda^{-1}_{(\mathbf{m})} \mathbf{m} \otimes \mathbf{m}^* + \lambda^{-1}_{(\mathbf{n})} \mathbf{n} \otimes \mathbf{n}^* \ . \tag{13}$$

For each given Ψ (or ϕ), we have the corresponding two by two decomposition

$$\mathbf{F}' = \mathbf{Q}'\mathbf{G}' = \mathbf{H}'\mathbf{Q}' \ , \tag{14}$$

where $\mathbf{Q}'$ is a two by two proper orthogonal tensor. It is a two by two "extended polar decomposition". The results are illustrated in the case of simple shear.

References

[1] Boulanger Ph. & Hayes M.: On finite shear. *Arch. Rational Mech. Anal.* **151**:125–185, 2000.

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[3] Truesdell C. & Toupin R.:The Classical Field Theories, Handbuch der Physik III/1. Springer Verlag, Berlin 1960.