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NONLINEAR WAVES IN ELASTIC SOLIDS

Wlodzimierz Domanski
Polish Academy of Sciences, IPPT PAN, Swietokrzyska 21, PL 00-049 Warszawa, Poland

Summary We derive evolution equations for the amplitudes of weakly nonlinear elastic waves. Some of these equations are new in the context of elastodynamics, like e.g. the complex Burgers equation describing the propagation of amplitudes of quasi-shear waves along the diagonal of a cube in a cubic crystal. New formulas are obtained for the wave interaction coefficients and then are used in the formulation of the condition which guarantees global existence of a classical solution to the initial-value problem of nonlinear elastodynamics equations.

INTRODUCTION

One of the open problems in continuum mechanics is to understand nonlinear dynamics of a deformable medium. Mathematical equations which describe macroscopic dynamics of a continuum are systems of balance laws, like e.g. balance of mass, momentum, energy etc. If we disregard dissipative and dispersive effects, these balance laws become, under natural assumptions, hyperbolic systems of conservation laws. Understanding the propagation and interaction of nonlinear waves, that is solutions of these nonlinear hyperbolic partial differential equations, is extremely important and challenging. Nonlinear waves, unlike linear, do not superimpose additively but instead may interact resonantly producing new waves which have frequencies and wave numbers being linear combinations of the frequencies and wave numbers of interacting waves. Besides, nonlinearity may cause formation of shock waves. Both of these nonlinear effects: *resonances* and *shocks* formation imply instability and create difficulties in analyzing the initial and initial-boundary value problems in continuum mechanics. In this paper we present some new results devoted to nonlinear waves in solids.

PROBLEM FORMULATION

Assumptions

We restrict ourselves to the *hyperelastic* continuum. Both *geometric* and *material* nonlinearities are taken into account. We assume that the system of elastodynamics equations is *hyperbolic*. It is convenient to write our equations as a first order system of partial differential equations:

$$\begin{aligned} \rho_0 \frac{\partial \boldsymbol{v}}{\partial t} &= \boldsymbol{Div}\, \boldsymbol{T}(\boldsymbol{F}), \\ \frac{\partial \boldsymbol{F}}{\partial t} &= \boldsymbol{Grad}\, \boldsymbol{v}, \end{aligned} \tag{1}$$

here ρ_0 is a constant density, $\boldsymbol{v}$ is a velocity, $\boldsymbol{T}$ is the first Piola-Kirchhoff stress, $\boldsymbol{F}$ is a deformation gradient tensor, and $\boldsymbol{Div}, \boldsymbol{Grad}$ denote the divergence and gradient operators in the reference configuration. The body forces are disregarded. We study the initial-value problem for the system (1) with different data and for different media. Both isotropic and some examples of anisotropic elastic solids are investigated.

Plane waves

It turns out that in many cases it is enough to look at plane wave solutions of (1). This allows to reduce the technical difficulties connected with the analysis of a 12×12 system (1) in three space dimension, to a 6×6 system (2) in one space dimension x (see e.g. [1] or [4]):

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k}) \frac{\partial \boldsymbol{w}}{\partial x} = \boldsymbol{0}, \tag{2}$$

where

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{v}(x,t) \\ \boldsymbol{m}(x,t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k}) = -\begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}(\boldsymbol{m}, \boldsymbol{k}) \\ \boldsymbol{I} & \boldsymbol{0}, \end{bmatrix} \tag{3}$$

$\boldsymbol{B}$ is a symmetric 3×3 matrix composed of second derivatives of the strain energy with respect to the components of the 1-D strain vector $\boldsymbol{m}$, and $\boldsymbol{k}$ is the direction of the wave propagation, $\boldsymbol{I}$ is a 3×3 identity matrix.
Due to the special block structure of the matrix $\boldsymbol{A}$ we may further reduce computational problems of finding the eigensystem of a 6×6 matrix $\boldsymbol{A}$ to the eigensystem of the reduced 3×3 matrix $\boldsymbol{B}$.

METHODS

Asymptotic expansion

We use the method of weakly nonlinear asymptotics to the initial-value problem for (2), around the constant state $\boldsymbol{w}_0$, in the form:

$$\boldsymbol{w}^{\epsilon}(x,t) = \boldsymbol{w}_0 + \epsilon \boldsymbol{w}_1(x,t,\eta) + \epsilon^2 \boldsymbol{w}_2(x,t,\eta) + \mathcal{O}(\epsilon^3) \tag{4}$$

with $\eta \equiv \epsilon^{-s}(x - \lambda t)$ being the phase variable, and the constant s determining appropriate scaling.
This is a multiple scales perturbation method which allows to derive nonlinear evolution equations for the waves amplitudes. We are particularly interested in nonclassical cases when *loss of strict hyperbolicity* and\or *loss of genuine nonlinearity* occur which happens in nonlinear elastodynamics. We analyze the evolution equations and we make conclusions about the behavior of the original elastodynamics system.

RESULTS

Evolution equations

We derive evolution equations which are new in the context of nonlinear elastodynamics. This is the case e.g. for a cubic crystal where the *complex Burgers equation* is a canonical equation for two pairs of quasi-shear waves propagating along the diagonal of a cube in a cubic crystal (see [1] or [5]). It can be written as a coupled system of two nonlinear equations with quadratic nonlinearity.

Interaction coefficients

The crucial role in the analysis of interaction of waves is played by the wave *interaction coefficients* Γ^j_{pq} (see [2] or [4]) which appear in the evolution equations. These coefficients describe which waves interact and also give information how strong the interaction is. The interaction coefficients are defined in terms of the left $\boldsymbol{l}_j$ and right $\boldsymbol{r}_j$ eigenvectors of matrix $\boldsymbol{A}$:

$$\Gamma^j_{pq} = \boldsymbol{l}_j \, \nabla_{\boldsymbol{w}} \boldsymbol{A}(\boldsymbol{w}) \, \boldsymbol{r}_p \, \boldsymbol{r}_q . \tag{5}$$

In our joint paper [4] with Robin Young we have derived new formulas for the coefficients (5) and expressed them entirely in terms of the derivatives of the strain energy. These new formulas allow us to calculate all the interaction coefficients for arbitrary elastic media. It turns out that the interaction coefficients may give information about stability or instability, as well as blow up or global existence of the original system of nonlinear elastodynamics.

Null condition

In another joint paper [3] with Ray Ogden we have investigated the so called *null condition* which implies global existence of a classical solution to the initial- value problem for the nonlinear elastodynamics equations. We were able to express this condition in terms of interaction coefficients and to analyze it for different types of elastic materials.

CONCLUSIONS

We have presented several new results which deal with solutions of nonlinear elastodynamics equations and help in understanding the problem of propagation and interaction of nonlinear elastic waves.

References

[1] Domanski W.: Weakly Nonlinear Elastic Plane Waves in a Cubic Crystal. *Contemp. Math.* **255**:45-61, 2000.
[2] Domanski W., Jablonski T.: On Resonances of Nonlinear Elastic Waves in a Cubic Crystal. *Arch. Mech.* **53**:91-104, 2001.
[3] Domanski W., Ogden R. W.: Null Condition for Nonlinear Elastic Materials. *to appear*.
[4] Domanski W., Young R.: Interaction of Plane Waves in Nonlinear Elasticity. *to appear*.
[5] Domanski W.: Asymptotic Equations for Weakly Nonlinear Elastic Shear Waves in a Cubic Crystal. *to appear*.