# ENERGY–CONSERVING SCHEME FOR NONLINEAR DYNAMICS OF SHELLS – NUMERICAL EXAMPLES

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*Summary* We propose an energy-conserving algorithm for dynamics of shell structures. The dynamic analysis is performed using the six-field non-linear shell model, with three independent translations and three rotation parameters. The spatial approximation is based on FEM. Our temporal integration procedure is based on mid-point approximation of the rate equations expressed in state variables. Numerical simulation of non-linear dynamics of the branched shell undergoing multiple turns and large relative deformations is presented.

## INTRODUCTION

The objective of the paper is to design an energy-conserving time-integration algorithm for the Hamiltonian dynamics of non-linear shell structures. Using such integration schemes plays the crucial role in the dynamic analysis of trusses, rods and shells as well as of multibody systems and more complex spatial structures. An efficient simulation of the overall motion of shell structures requires the use of relatively long time steps, and consequently the application of rather implicit time-stepping schemes. The FEM solution of flexible irregular shell structures leads to the large set of stiff equations. It can reveal a wide spectrum of vibration frequencies. In such cases some standard time-integration schemes loose their stability. This manifests itself, among others, by the sudden increase of the total energy, see [1,2], and simulations cannot be continued without reduction of the time step size. One of the ways to overcome this problem is to apply energy-conserving time-integration schemes. A number of algorithms capable of enforcing the energy conservation are proposed, e.g. [3–10].

## ENERGY–CONSERVING INTEGRATION SCHEME

In this paper we propose an energy-conserving algorithm for shell dynamics based on the six-field shell model [11,14], that is with six engineering degrees of freedom in each FE node [12]. Motion of the irregular shell structure is described by the displacement vector field $\boldsymbol{u}=\boldsymbol{y}-\boldsymbol{x}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are position vectors of the undeformed $M$ and deformed shell base surface, and by the proper orthogonal tensor field $\boldsymbol{Q}\in SO(3)$. As a result, the rotation group enters the definition of the configuration space. In such problems of structural mechanics finite-dimensional approximations, like the finite element method and the temporal approximation, require non-standard approaches (see e.g. [1,12–14]). Kinetic constitutive relations for the translational and rotational momentum vectors are $\boldsymbol{p}=m_0\boldsymbol{v}$ and $\boldsymbol{\pi}=I_0\boldsymbol{\omega}$ (see [11]), where $\boldsymbol{v}=\dot{\boldsymbol{y}}$ and $\boldsymbol{\omega}$ are the linear and angular velocity vectors in the spatial representation, with $\mathrm{ad}\boldsymbol{\omega}=\dot{\boldsymbol{Q}}\boldsymbol{Q}^T$, $\mathrm{ad}: E^3\to so(3)$. The initial-boundary value problem in the weak form for the general dynamics of branched shell structures was formulated in [1,2,13]. For hyperelastic shells there exists a 2D strain energy density $W(\boldsymbol{\varepsilon}_\beta,\boldsymbol{\kappa}_\beta;\boldsymbol{x})$, with the shell strain vectors $\boldsymbol{\varepsilon}_\beta=\boldsymbol{y}_{,\beta}-\boldsymbol{Q}\boldsymbol{x}_{,\beta}$ and $\boldsymbol{\kappa}_\beta=\mathrm{ad}^{-1}(\boldsymbol{Q}_{,\beta}\boldsymbol{Q}^{\mathrm{T}})$, which implies the constitutive relations $\boldsymbol{n}^\beta=\partial W/\partial\boldsymbol{\varepsilon}_\beta$, $\boldsymbol{m}^\beta=\partial W/\partial\boldsymbol{\kappa}_\beta$, where $\boldsymbol{n}^\beta$ and $\boldsymbol{m}^\beta$ are the internal stress and couple resultant vectors, respectively.

The energy-conserving integration scheme of Simo and Tarnow [3] was applied to dynamics of shells e.g. in [4–7]. However, in dynamics of the six-field shell structures more adequate seems to be the algorithm proposed for rods in [8] and extended e.g. in [9,10]. The algorithmic energy-conserving approximation in the weak form is written here not in terms of acceleration, but in terms of difference of the momenta at $t_{\mathrm{n+1}}$ and $t_{\mathrm{n}}$ instants:

$$\frac{1}{\Delta t}\iint_{M\setminus\Gamma}[(\boldsymbol{p}_{n+1}-\boldsymbol{p}_n)\bullet\boldsymbol{v}+(\boldsymbol{\pi}_{n+1}-\boldsymbol{\pi}_n)\bullet\boldsymbol{w}]\,da+\iint_{M\setminus\Gamma}[\boldsymbol{n}^\beta\bullet(\boldsymbol{v}_{,\beta}+\boldsymbol{y}_{n+1/2,\beta}\times\boldsymbol{w})+\boldsymbol{m}^\beta\bullet\boldsymbol{w}_{,\beta}]\,da=G_{\mathrm{ext}}[\boldsymbol{u},\boldsymbol{Q};\boldsymbol{v},\boldsymbol{w}]\,, \tag{1}$$

where $\boldsymbol{y}_{n+1/2}=\frac{1}{2}(\boldsymbol{y}_{n+1}+\boldsymbol{y}_n)$, and $\boldsymbol{v}$, $\boldsymbol{w}$ are the virtual vector fields. In this algorithm the Cayley transformation $\Delta\mathbf{Q}=\mathrm{cay}(\Delta\mathbf{C})=\boldsymbol{I}+(1+\frac{1}{4}\Delta\mathbf{c}\bullet\Delta\mathbf{c})^{-1}(\Delta\mathbf{C}+\frac{1}{2}\Delta\mathbf{C}^2)=(\boldsymbol{I}+\frac{1}{2}\Delta\mathbf{C})(\boldsymbol{I}-\frac{1}{2}\Delta\mathbf{C})^{-1}$, $\boldsymbol{Q}_{n+1}=\boldsymbol{Q}_n\Delta\mathbf{Q}$, $\Delta\mathbf{C}=\mathrm{ad}(\Delta\mathbf{c})$, plays the crucial role. It allows one to formulate the mid-point approximation of the rate equations expressed in state variables as follows:

$$\boldsymbol{y}_{n+1}-\boldsymbol{y}_n=\boldsymbol{u}_{n+1}-\boldsymbol{u}_n=\Delta\boldsymbol{u}\,,\qquad \boldsymbol{Q}_{n+1}-\boldsymbol{Q}_n=\tfrac{1}{2}(\boldsymbol{Q}_{n+1}+\boldsymbol{Q}_n)\,\Delta\mathbf{C}=\mathbf{Y}_{n+1/2}\Delta\mathbf{C}\,, \tag{2}$$

where $\Delta\boldsymbol{u}=\frac{1}{2}\Delta t(\boldsymbol{v}_{n+1}+\boldsymbol{v}_n)$, $\Delta\mathbf{c}=\frac{1}{2}\Delta t(\boldsymbol{\omega}_{n+1}+\boldsymbol{\omega}_n)$, $\boldsymbol{\omega}=\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\omega}$, $\mathbf{Y}_{n+1/2}=\frac{1}{2}(\boldsymbol{Q}_{n+1}+\boldsymbol{Q}_n)$. Moreover, the tensor $\mathbf{Y}_{n+1/2}\notin SO(3)$ and the so-called algorithmic constitutive equations in the material representation are the key points of the formulation. The linearised equations of shell dynamics of the type (1) are written in the spatial representation, while the rotation,

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angular velocity and acceleration fields are computed in the material representation. Such an algorithm possesses the energy conservation property while retaining the convenience and efficiency of conventional time-stepping schemes.

## NUMERICAL EXAMPLES

Large overall motion of the flexible cylindrical panel with change of the curvature sign, reinforced by a plate rib (see Fig. 1), is analysed. The pair of forces, described by the ramp time function, is initially applied to the structure during a short time period. After two seconds the shell is free from external loading and moves freely in space.

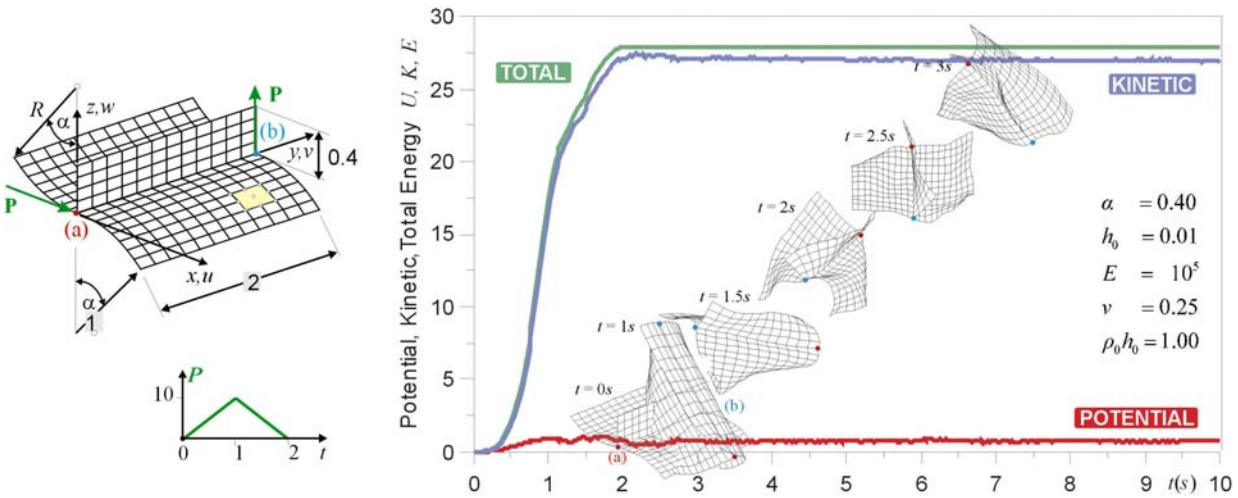



Fig. 1 Cylindrical panel. Geometry, energy functions and shell deformations

The example has been chosen for two reasons. Firstly, the geometry of the structure sharply verifies accuracy and capability of the shell theory applied, like e.g. good description of orthogonal connection of the shell branches. It is also a demanding test for importance of the sixth degree of freedom. Secondly, the overall motion of the shell is also an adequate test verifying the time-stepping scheme. The integration was done with the time step $\Delta t = 0.02\,s$. In the spatial discretization we used 9- and 16-node finite element (FI) with six degrees of freedom in each node. In both cases the energy levels were close to each other. The structure moves freely in space undergoing multiple turns and large relative deformations. Several stages of the motion and the energy histories are shown in Fig. 1.

Following the papers [15] and [16], in the next example, we consider the dynamics of three intersecting plates. The ramping load is applied to the system as in the example above. The shell geometry and resultant forces of the load uniformly distributed along the shell edges are depicted in Fig. 2. Several stages of the motion with shell deformations are shown in Fig. 3.

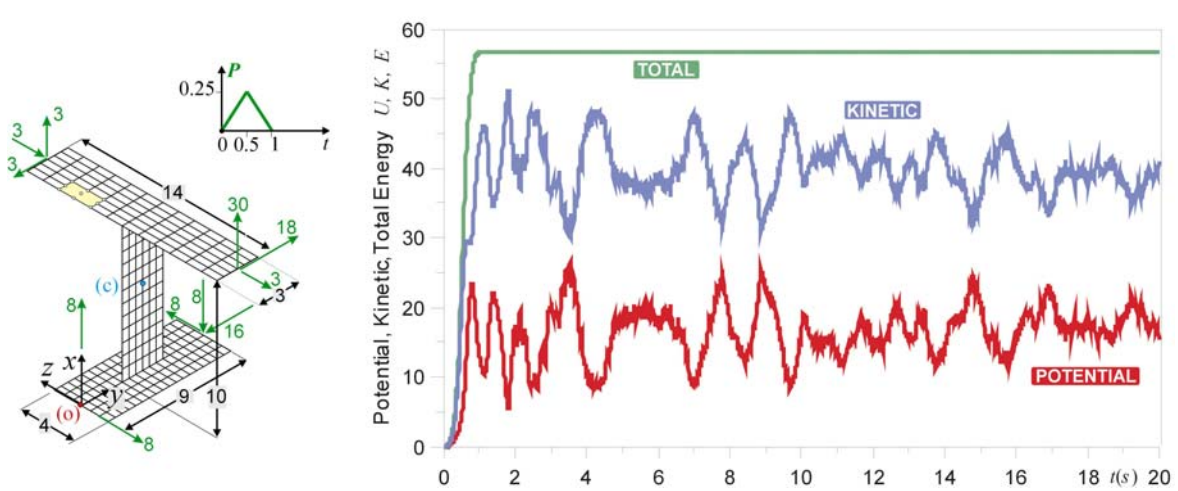
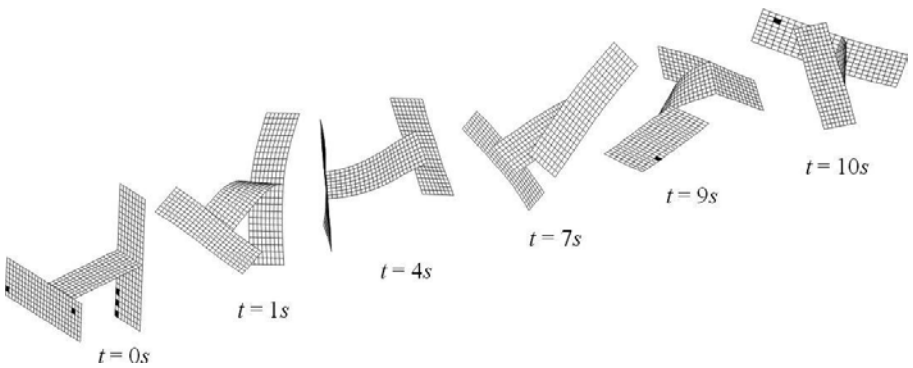



Fig. 2 Three intersecting plates. Geometry and energy functions

In the analysis the time step $\Delta t = 0.002\,s$ was used. Fig. 2 presents the results obtained with 9-node finite elements (FI) with mesh $4\times12$ elements for each plate. The computed total energy (~56), presented in the figure above, is different than that reported in references [15] and [16]: ~70 and ~20 respectively. This may be attributed, probably, to different

material data, which are not clearly depicted in both papers, and/or some problems with the convergence (mesh and locking phenomenon) of the finite element approximation. This issue has been already addressed in [17].




$$E = 2\times10^{7}$$
$$\nu = 0.25$$
$$h_0 = 0.02$$
$$J_{\rho_0} = 3.333\times10^{-5}$$
$$m_{\rho_0} = 0.02$$

Fig. 3 Three intersecting plates. Data and shell deformations

In this example, the use of 16-node finite elements (FI) with the same mesh yields an increase of the total energy (~60). This may stem from the second fact mentioned above.

## CONCLUSIONS

The analysis presented in [1,12,13], based on the classical extended Newmark scheme, after some time leads to unjustified increase of the total energy of the system, which finally leads to numerical instability. However, application of our energy-conserving integration scheme gives in the first example the same results as those in [1,2,13] in the stable range and does not reveal instabilities. The second example shows that the energy-conserving scheme assures the stability of the solution within the entire range (of 20*s*) under investigation (Fig. 2). The comparable results from [15] and [16] are presented in a 10*s* and 15*s* period respectively.

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