H-CONVERGENCE AND MULTILAYERING IN PIEZOCOMPOSITES

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Summary The aim of this paper is to solve the following problems related to piezocomposite: 1) to extend the notion of H-convergence 2) to derive the formula for the effective moduli of multilaminated composite. The obtained formula for the lamination is used to optimal design for minimum compliance.

EXTENSION OF H-CONVERGENCE TO PIEZOELECTRICITY

Microstructure of piezoelectric material is characterized by a small parameter ε, say $\varepsilon = 1/n$, where $n \in \mathrm{N}$; N stands for the set of natural numbers. Homogenization means passing with ε to 0. Physically such a passage denotes smearing out microinhomogeneities. The considered piezoelectric body occupies $\Omega \subset \mathbb{R}^3$ a bounded, sufficiently regular domain, $\Gamma = \partial\Omega$, $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma = \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, $\Gamma_2 \cap \Gamma_3 = \emptyset$. Let U^ε denote the internal energy for fixed $\varepsilon > 0$. It has the following form: $U^\varepsilon(\mathbf{e},\mathbf{D}) = \frac{1}{2} a^\varepsilon_{ijkl} e_{ij} e_{kl} - h^\varepsilon_{ijk} D_i e_{jk} + \frac{1}{2}\kappa^\varepsilon_{ij} D_i D_j$

where the strain $e \in M^s_3$ and electric induction $\mathbf{D} \in \mathbb{R}^3$. By M^s_3 we denote the set of symmetric 3 x 3 matrices. The electric enthalpy $H^\varepsilon(\mathbf{e},\mathbf{E})$ is calculated as partial concave conjugate of $U^\varepsilon(\mathbf{e},\mathbf{D})$,

$$H^\varepsilon(\mathbf{e},\mathbf{E}) = \inf\left\{-\mathbf{E}\cdot\mathbf{D} + U^\varepsilon(\mathbf{e},\mathbf{D}) \,|\, \mathbf{D} \in \mathrm{R}^3\right\} = \frac{1}{2} c^\varepsilon_{ijkl} e_{ij} e_{kl} - g^\varepsilon_{ijk} E_i e_{jk} - \frac{1}{2} \epsilon^\varepsilon_{ij} E_i E_j,$$

where $\mathbf{E}$ = -grad φ; φ is the electric potential, $c^\varepsilon_{ijkl} = a^\varepsilon_{ijkl} - g^\varepsilon_{mij} h^\varepsilon_{mkl}$, $\epsilon^\varepsilon = (\kappa^\varepsilon)^{-1}$, $g^\varepsilon_{mij} = \epsilon^\varepsilon_{ml} h^\varepsilon_{lij}$.

We observe that under physically plausible assumptions the internal energy U^ε is convex whilst H^ε is concave in $\mathbf{E}$. We assume the material moduli are essentially bounded, i.e., they belong to $L^\infty(\Omega)$. We set

$$\mathbf{A}^\varepsilon = \begin{bmatrix} \mathbf{c}^\varepsilon & -\mathbf{g}^\varepsilon \\ \mathbf{g}^\varepsilon & \epsilon^\varepsilon \end{bmatrix}, \qquad \mathbf{B}^\varepsilon = \begin{bmatrix} \mathbf{a}^\varepsilon & -\mathbf{h}^\varepsilon \\ -\mathbf{h}^\varepsilon & \boldsymbol{\kappa}^\varepsilon \end{bmatrix},$$

The constitutive relationship takes the following form:

$$\sigma^\varepsilon_{ij} = \frac{\partial H^\varepsilon}{\partial e^\varepsilon_{ij}} = c^\varepsilon_{ijkl} e^\varepsilon_{kl} - g^\varepsilon_{kij} E^\varepsilon_k \qquad \sigma^\varepsilon_{ij} = \frac{\partial U^\varepsilon}{\partial e^\varepsilon_{ij}} = a^\varepsilon_{ijkl} e^\varepsilon_{kl} - h^\varepsilon_{kij} D^\varepsilon_k$$

or equivalently

$$D^\varepsilon_i = -\frac{\partial H^\varepsilon}{\partial E^\varepsilon_i} = g^\varepsilon_{ikl} e^\varepsilon_{kl} + \epsilon^\varepsilon_{ik} E^\varepsilon_k \qquad E^\varepsilon_i = \frac{\partial U^\varepsilon}{\partial D^\varepsilon_i} = -h^\varepsilon_{ikl} e^\varepsilon_{kl} + \kappa^\varepsilon_{ik} D^\varepsilon_k.$$

Let $\{\mathbf{B}^\varepsilon(\mathbf{x})\}_{\varepsilon>0}$, $\mathbf{x}\in\Omega$, be a sequence of matrices such that $\mathbf{B}^\varepsilon \in L^\infty(\Omega; M_{\alpha,\beta})$, where $\beta \geq \alpha > 0$ denote constants and

$$M_{\alpha,\beta} = \{\mathbf{B} \mid a_{ijlk} \in L^\infty(\Omega), h_{ijk} \in L^\infty(\Omega), \kappa_{ij} \in L^\infty(\Omega), \mathbf{B}(\mathbf{x})\phi\cdot\phi \geq \alpha(|\phi_1|^2 + |\phi_2|^2),$$

$\mathbf{B}^{-1}(\mathbf{x})\phi\cdot\phi \geq \beta(|\phi_1|^2 + |\phi_2|^2)$, $\forall \phi_1 \in M^s_3, \phi_2 \in R^3$, for almost every $\mathbf{x} \in \Omega\}$. It is clear that piezoelectric layered materials are covered by the definition of $M_{\alpha,\beta}$.

A sequence of matrices $\mathbf{B}^\varepsilon(\mathbf{x}) \in L^\infty(\Omega; M_{\alpha,\beta})$ is **H-convergent** to $\mathbf{B}^h \in L^\infty(\Omega; M_{\alpha,\beta})$ if for arbitrary $b_i \in H^{-1}(\Omega)$, $f_i \in H^{-1/2}(\Gamma_1)$, $d \in H^{-1/2}(\Gamma_3)$, $\varphi_0 \in H^{1/2}(\Gamma_2)$, the sequence $(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon)_{\varepsilon>0}$ of solutions to the equations

$-[a^\varepsilon_{ijkl} e^\varepsilon_{kl}(\mathbf{u}^\varepsilon) - h^\varepsilon_{kij} D^\varepsilon_k]_{,j} = b_i$, in Ω $\qquad D^\varepsilon_{i,i} = 0$, $\varepsilon_{lim}[h^\varepsilon_{ijk} e_{jk}(\mathbf{u}^\varepsilon) + \kappa^\varepsilon_{ij} D^\varepsilon_j]_{,m} = 0$ in Ω

with the boundary conditions $u^\varepsilon_i = 0$ on Γ_0, $\sigma^\varepsilon_{ij} n_j = f_i$ on Γ_1, $\varphi^\varepsilon = \varphi_0$ on Γ_2, $D^\varepsilon_i n_i = d$ on Γ_3.

is such that $\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}$ weakly in V, $\mathbf{D}^\varepsilon \rightharpoonup \mathbf{D}$ weakly in $L^2(\Omega)^3$, $a^\varepsilon_{ijkl} e^\varepsilon_{kl}(\mathbf{u}^\varepsilon) - h^\varepsilon_{kij} D^\varepsilon_k \rightharpoonup a^h_{ijkl} e_{kl}(\mathbf{u}) - h^h_{kij} D_k$ weakly in V^*, $-h^\varepsilon_{ijk} e_{jk}(\mathbf{u}^\varepsilon) + \kappa^\varepsilon_{ij} D^\varepsilon_j \rightharpoonup -h^h_{ijk} e_{jk}(\mathbf{u}) + \kappa^h_{ij} D_j$ weakly in $L^2(\Omega)$ where $(\mathbf{u},\mathbf{D})$ is a solution to the homogenized system

$[a^h_{ijkl} e_{kl}(\mathbf{u}) - h^h_{kij} D_k]_{,j} = b_i$, in Ω, $\quad D_{i,i} = 0$, $\varepsilon_{lim}[h^h_{ijk} e_{jk}(\mathbf{u}) + \kappa^h_{ij} D^h_j]_{,m} = 0$ in Ω,

$u_i = 0$ on Γ_0, $\quad \sigma_{ij} n_j = f_i$ on Γ_1, $\quad \varphi = \varphi_0$ on Γ_2, $\quad D_i n_i = d$ on Γ_3.

Here $V = \left\{ v \in H^1(\Omega)^3 \,\middle|\, v = 0 \text{ on } \Gamma_0 \right\}$ and V^* is the dual space of V; moreover $\sigma^h_{ij} = a^h_{ijkl} e_{kl} - h^h_{kij} D_k$,

$E^h_i = h^h_{ijk} e_{jk}(\mathbf{u}) + \kappa^h_{ij} D^h_j$.

MULTILAYERING

The constitutive relationship for piezocomposite can be written as follows

$$\mathbf{\Sigma}^{\varepsilon} = \mathbf{A}^{\varepsilon}\mathcal{E}^{\varepsilon}, \text{ where } \quad \mathbf{\Sigma}^{\varepsilon} = \begin{bmatrix} \boldsymbol{\sigma}^{\varepsilon} \\ \mathbf{D}^{\varepsilon} \end{bmatrix}, \quad \mathbf{A}^{\varepsilon} = \begin{bmatrix} \mathbf{c}^{\varepsilon} & -\mathbf{g}^{\varepsilon} \\ \mathbf{g}^{\varepsilon} & \in^{\varepsilon} \end{bmatrix}, \quad \mathcal{E}^{\varepsilon} = \begin{bmatrix} \mathbf{e}^{\varepsilon} \\ \mathbf{E}^{\varepsilon} \end{bmatrix}, \quad \mathbf{E}^{\varepsilon} = -\mathrm{grad}\varphi^{\varepsilon}.$$

Let us now assume that the moduli depend only on one variable in the direction of unit vector $\mathbf{n}_1$, e.g. $\mathbf{n}_1$ denotes the unit vector of axis x_1: $\mathbf{A}^{\varepsilon} = \chi^{\varepsilon}(x_1)\overset{(1)}{\mathbf{A}} + (1-\chi^{\varepsilon}(x_1))\overset{(2)}{\mathbf{A}}$, where $\overset{(1)}{\mathbf{A}}$, $\overset{(2)}{\mathbf{A}}$ are the matrices of materials (1) and (2), respectively. Here $\chi^{\varepsilon}(x_1)$ is the characteristic function equal 1 in the domain occupied by the material (1) and equal zero otherwise.

Performing homogenization (H-convergence), i.e., letting ε tend to zero we get the macroscopic constitutive relationship in the form:

$$\mathbf{\Sigma} = \mathbf{A}^{h}\,\mathcal{E}, \quad \text{where} \quad \mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \end{bmatrix}, \quad \mathbf{A}^{h} = \begin{bmatrix} \mathbf{c}^{h} & -\mathbf{g}^{h} \\ \mathbf{g}^{h} & \in^{h} \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} \mathbf{e} \\ \mathbf{E} \end{bmatrix},$$

We denote by θ the weak-* limit of the sequence of the characteristic functions $\{\chi^{\varepsilon}\}$, i.e., $\chi^{\varepsilon} \rightharpoonup \theta$ in $L^{\infty}(\Omega)$ as ε tends to zero.
The following implicit formula for the effective piezoelectric moduli of layered composite:

$$\theta(\mathbf{A}^{h} - \overset{(2)}{\mathbf{A}})^{-1}\boldsymbol{\eta} = (\overset{(1)}{\mathbf{A}} - \overset{(2)}{\mathbf{A}})^{-1}\boldsymbol{\eta} + (1-\theta)(\mathbf{q}_{(2)}(\mathbf{n}_1)(\boldsymbol{\eta}\mathbf{n}_1)) \oplus \mathbf{n}_1$$

where $\mathbf{q}^{-1}_{(2)}(\mathbf{n}_1)$ is a symmetric 4x4 matrix given by $\mathbf{q}^{-1}_{(2)}(\mathbf{n}_1) = (\begin{bmatrix} \overset{(2)}{c}_{ijkl}\, n_{1j}n_{1l} & -\overset{(2)}{g}_{ijk}\, n_{1j}n_{1k} \\ \overset{(2)}{g}_{jkl}\, n_{1l}n_{1j} & \overset{(2)}{\in}_{ij}\, n_{1i}n_{1j} \end{bmatrix})$.

The relationship for the pth-order lamination, performed as follows. In the first step we find the homogenized matrix $\mathbf{A}^{h}_{1}$ for a piezoelectric layered two-phase material characterized by $\overset{(1)}{\mathbf{A}}, \overset{(2)}{\mathbf{A}}$, the volume fraction θ_1 and lamination direction $\mathbf{n}_1$. In the second step of lamination we find the homogenized matrix $\mathbf{A}^{h}_{2}$ for a layered two-phase piezoelectric material characterized by $\mathbf{A}^{h}_{1}$, $\overset{(2)}{\mathbf{A}}$, the volume fraction θ_2 and lamination direction $\mathbf{n}_2$, etc. Finally at the p-th step we get

$$\prod_{i=1}^{p}\theta_i(\mathbf{A}^{h}_{p} - \overset{(2)}{\mathbf{A}})^{-1}\boldsymbol{\eta} = (\overset{(1)}{\mathbf{A}} - \overset{(2)}{\mathbf{A}})^{-1}\boldsymbol{\eta} + \sum_{i=1}^{p}(1-\theta_i)\left[(\mathbf{q}_{(2)}(\mathbf{n}_i)(\boldsymbol{\eta}\mathbf{n}_i)) \oplus \mathbf{n}_i\right]\prod_{k=1}^{i-1}\theta_k$$

FINAL REMARKS

We conclude that the derived formula for the lamination is convenient for a control of parameters involved in it. For instance, changing angles of lamination and preserving the volume fractions of successive steps of the lamination we can generate composites with various properties. Similarly, preserving directions and choosing optimal, from prescribed point of view partial volume fractions we have wide possibilities of control of macroscopic properties of composites.

An essential role in the study of optimal design for minimum compliance is played by homogenization and relaxation of relevant compliance functional. More precisely, it has been shown that a two-phase composite with minimum compliance (maximum stiffness) is realized by laminates of suitable order. This deep result has been proved only for composites made of two isotropic elastic materials with order bulk and shear moduli, see [1,2].

The compliance functional is assumed in the following form

$$J(\chi) = \int_{\Omega} b_i u_i dx + \int_{\Gamma_1} f_i u_i d\Gamma + \int_{\Gamma_2} D_i n_i \varphi_0 d\Gamma + \int_{\Gamma_3} \bar{d}\varphi d\Gamma.$$

The minimum compliance problem means evaluating

$$\inf\{\, J(\chi) | \chi \in L^{\infty}(\Omega;\{0,1\}), \quad \int_{\Omega}\chi(\mathbf{x})d\mathbf{x} = V_1, \quad \text{equilibrium and boundary conditions are satisfied}\}.$$

Similarly to the elastic case one can consider optimal shape design for minimum compliance, cf. [1,2]. Formally, it suffices to assume that the moduli of weaker material disappear. However, it is worth noting that a piezocomposite with optimal shape may consists of three regions; the first region is made of material (1) (the stronger one), the second region contains voids whilst the third region is a composite obtained by mixing (in homogenization sense) of the material (1) and microvoids.

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References

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