STABILITY OF PARAMETRICALLY EXCITED STRUCTURES: NEW RESULTS

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Summary Linear dynamical systems with many degrees of freedom with periodic coefficients also depending on constant parameters are considered. Stability of the trivial solution is studied with the use of the Floquet theory. First and second order derivatives of the Floquet matrix with respect to parameters are derived in terms of matriciants of the main and adjoint problems and derivatives of the system matrix. It is shown how to use this information in gradient procedures for stabilization or destabilization of the system. Then, linear vibrational systems with periodic coefficients depending on three independent parameters: frequency and amplitude of periodic excitation, and damping parameter are considered with the assumption that the last two quantities are small. For arbitrary matrix of periodic excitation and positive definite damping matrix general expressions for regions of the main (simple) and combination resonances are derived. Two important specific cases of excitation matrix are studied: a symmetric matrix and a stationary matrix multiplied by a scalar periodic function. It is shown that in both cases the resonance regions are halves of cones in the three-dimensional parameter space with the boundary surface coefficients depending only on the eigenfrequencies, eigenmodes of the conservative system and system matrices. As an example of the developed theory Bolotin's problem on dynamic stability of a beam loaded by periodic bending moments is solved.

DERIVATIVES OF THE FLOQUET MATRIX WITH RESPECT TO PARAMETERS

We consider a system of linear differential equations

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}, \tag{1}$$

where $\mathbf{G} = \mathbf{G}(t, \mathbf{p})$ is a real square matrix of dimension m, which is smoothly depending on a vector of real parameters $\mathbf{p} = (p_1, p_2, ... p_n)$ and is a continuous periodic function of the time $\mathbf{G}(t, \mathbf{p}) = \mathbf{G}(t+T, \mathbf{p})$, T being a period. We denote linear independent solutions of system (1) as $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)$ and form out of them a fundamental matrix $\mathbf{X}(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)]$. The $\mathbf{X}(t)$ matrix satisfying the equations

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}, \tag{2}$$

where $\mathbf{I}$ is the identity matrix of dimension m, is called a matriciant, and the matrix $\mathbf{F} = \mathbf{X}(T)$ is called a monodromy matrix.

According to the Floquet theory, see Nayfeh and Mook [1], stability of system (1) is determined by multipliers (eigenvalues of the monodromy matrix): if all the multipliers for their absolute value are less than one, the system is asymptotically stable, and if at least one of them is greater than one, the system becomes unstable.

Now we assume that the vector of parameters takes a variation $\mathbf{p} = \mathbf{p_0} + \Delta\mathbf{p}$. Hence, the $\mathbf{G}$ matrix, and therefore the matriciant $\mathbf{X}(t)$ obtain variations. This accordingly leads to a change of the monodromy matrix $\mathbf{F}$. The formulas for the first and second derivatives of a monodromy matrix with respect to parameters are derived in the form of integrals over the period

$$\frac{\partial \mathbf{F}}{\partial p_k} = \mathbf{F_0} \int_0^T \mathbf{X_0}^{-1} \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X_0}\, dt, \quad k = 1, ..., n, \tag{3}$$

$$\frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} = \mathbf{F_0} \{ \int_0^T \mathbf{X_0}^{-1} \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} \mathbf{X_0}\, dt + \int_0^T \mathbf{X_0}^{-1} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X_0} \left(\int_0^\tau \mathbf{X_0}^{-1} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X_0}\, d\zeta \right) d\tau +$$
$$+ \int_0^T \mathbf{X_0}^{-1} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X_0} \left(\int_0^\tau \mathbf{X_0}^{-1} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X_0}\, d\zeta \right) d\tau \}, \quad i, j = 1, ..., n, \tag{4}$$

where the zero subscript means that the corresponding value is taken at $\mathbf{p} = \mathbf{p_0}$. Note that to find derivatives (3) and (4) it is necessary to know only the matriciants $\mathbf{X_0}(t)$ and the derivatives of the $\mathbf{G}$ matrix with respect to the parameters taken at $\mathbf{p} = \mathbf{p_0}$. Using derivatives (3) and (4) a variation of the monodromy matrix can be given in the form

$$\mathbf{F}(\mathbf{p_0} + \Delta\mathbf{p}) = \mathbf{F_0} + \sum_{k=1}^{n} \frac{\partial \mathbf{F}}{\partial p_k} \Delta p_k + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} \Delta p_i \Delta p_j + \cdots \tag{5}$$

Knowing the derivatives of the monodromy matrix we can calculate the value of this matrix in the vicinity of the initial point $\mathbf{p_0}$, and therefore estimate behavior of the multipliers responsible for the stability of system (1) when the problem parameters are changed. This is what we call sensitivity analysis of multipliers. Analysis of multipliers provides information for determining stability and instability regions in the parameter space.

MECHANICAL EXAMPLE

As an example of the developed theory, we consider Bolotin's problem on dynamic stability of the trivial solution (plane position) of a beam, see Bolotin [2]. The elastic beam is assumed to be simply supported at its ends and loaded by the periodic bending moments $M(\Omega t) = \delta\varphi(\Omega t)$ in the plane of its maximum stiffness, where $\varphi(t)$ is a 2π -periodic function, Fig. 1. Bending-torsional vibrations off this plane are described by the equations

$$m\frac{\partial^2 w}{\partial t^2} + \gamma m d_1 \frac{\partial w}{\partial t} + EJ\frac{\partial^4 w}{\partial x^4} + \delta\varphi(\Omega t)\frac{\partial^2 \theta}{\partial x^2} = 0$$
$$mr^2\frac{\partial^2 \theta}{\partial t^2} + \gamma m r^2 d_2 \frac{\partial \theta}{\partial t} + \delta\varphi(\Omega t)\frac{\partial^2 w}{\partial x^2} - GI\frac{\partial^2 \theta}{\partial x^2} = 0 \qquad (6)$$

Here, $w(x,t)$ is the transverse deflection of the beam; $\theta(x,t)$ and *r* are the torsion angle and the radius of inertia for the beam's cross section, respectively; *EJ* and *GI* are the bending and torsion stiffnesses of the beam, respectively; *m* is the mass per unit length of the beam; γ is the parameter of dissipative force (viscous friction coefficient); and d_1 and d_2 are fixed constants defining the bending and torsional viscous friction forces.

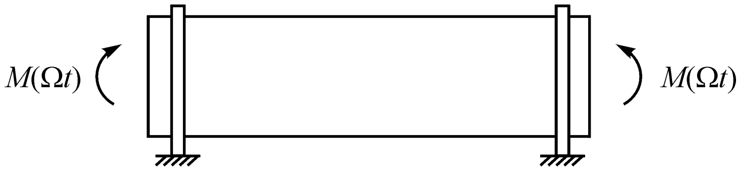

Figure 1. An elastic beam loaded by periodic bending moments.

The boundary conditions take the form

$$x = 0,\ l: \quad w = \frac{\partial^2 w}{\partial x^2} = \theta = 0 \qquad (7)$$

where *l* is the beam length. The problem consists in finding parametric resonance domains in the three-dimensional parameter space δ, Ω, and γ. It turns out that the regions of the difference-type combination resonance are empty, and only summation-type resonance takes place. The regions of the summation-type combination resonance at frequencies close to $\Omega_0 = (\omega_{n1} + \omega_{n2})/k$, *k*=1,2,... are determined by the formula

$$d_1 d_2 \gamma^2 - \frac{\pi^4 n^4}{l^4 r^2 m^2}\frac{(a_k^2 + b_k^2)}{4\omega_{n1}\omega_{n2}}\delta^2 + 4k^2\frac{d_1 d_2}{(d_1 + d_2)^2}\Delta\Omega^2 \le 0 \qquad (8)$$

where a_k and b_k are the Fourier coefficients of the periodic function $\varphi(t)$, and ω_{n1}, ω_{n2} are the eigenfrequencies of the conservative (undamped) system. Other mechanical examples including determination of the parametric resonance regions of elastic columns of non-uniform cross-section loaded by longitudinal periodic forces, and column optimization problems are presented in Seyranian, Solem and Pedersen [3], Seyranian [4], and Mailybaev and Seyranian [5]. Extended discussion of periodically excited structures and mechanical systems is given in a recent book by Seyranian and Mailybaev [6].

References

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