An exactly solvable microgeometry in torsion

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Summary Finding a geometric configuration that is amenable to an exact characterization of the torsional rigidity is a relatively new territory that has only recently begun to be explored. Here we present our latest finding of an exactly solvable microgeometry in torsion for cylindrical shafts with arbitrary cross sections.

Torsional rigidity is a crucial physical property in characterizing the twisting stiffness of a cylindrical object. This quantity, defined as the proportional ratio between the applied torque and the relative rotation of a cylindrical shaft, is generally understood as a structural property rather than a material property. Particularly, the rigidity depends not only on the microstructure of the cross section but also on its global geometry. Thus, the task of a general characterization of the torsional rigidity of a composite shaft, applicable for a variety of cross sections, has been hardly conceivable. A recent attempt [1] along this line was the finding of an exact torsional rigidity for a circular shaft that is filled with composite cylinder assemblage (CCA). The finding of the exact torsional rigidity of the CCA microgeometry in a circular cross section was striking in that it represents that the CCA microgeometry also permits an exact result under twisting loads. In this study, we make a further attempt by allowing that the cross section of the host shaft could be arbitrary in shape [2]. We show that, depending on the cross-sectional shape of the host shaft, an assemblage of multicoated cylinders is an exactly solvable microgeometry. To start with, we consider a circular multicoated cylinder, which is to be embedded inside the host shaft at a selected location. Under an applied torque at the ends of the cylindrical shaft, we ask what is the suitable shear modulus of the host shaft μ_0, or equivalently, what are the relationships between the phase moduli and area fractions of the multicoated inclusion, so that after the insertion of the multicoated cylinder inside the host shaft, the warping field and also shear traction in the shaft, remains unchanged as that of the unreinforced homogeneous shaft. Remarkably, we find that the framework of finding a neutral multicoated inclusion in a cylindrical shaft of arbitrary cross section in torsion is mathematically analogous to that of the conductivity problem but subject to boundary conditions of various orders. For instance, for a circular host shaft, it is equivalent to boundary conditions with positions of first order. For an elliptical host shaft, it corresponds to boundary conditions with positions of second order. When the cross section of the shaft becomes more irregular, e.g. rectangles, the order of boundary terms will become larger or even unbounded. For the present context in which the boundary condition contains various orders of positions, no relevant solutions were reported before. Nevertheless, we find that, depending on the presence of the orders of boundary terms, constraint conditions in the form of continued fraction can be established. For example, for a doubly coated cylinder (three-phase) with boundary terms of order N, exactly N constraint conditions need to be fulfilled

$$\mu_0 = \mu_1 + \cfrac{2\left(1-\nu_1^{(m)}\right)\mu_1}{\nu_1^{(m)} - \cfrac{2\mu_1}{\mu_1 - \mu_2 - \cfrac{2\nu_3^{(m)}\mu_2}{\nu_2^{(m)} - \cfrac{2\left(1-\nu_1^{(m)}\right)\mu_2}{\mu_2-\mu_3}}}}, \text{ with } m = 1, \cdots, N. \tag{1}$$

For a multicoated inclusion, we will demonstrate how to generate these continued fraction expressions of the constraint conditions without further derivations. Note that for an n-phase multicoated inclusion, we have $2n-1$ adjustable parameters for N constraint conditions. Conceptually, for a large N, one can generalize the multicoated inclusion as a functionally graded inclusion to have sufficient adjustable parameters to fit the constraint conditions. We mention that the constraint conditions are nonlinear algebraic equations. Finding exact analytical parameters that fulfill the constraint conditions may not be a simple task, but there is no difficulty to resolve the admissible sets of parameters numerically. Our next task is to investigate the torsional rigidity of a shaft containing a neutral multicoated inclusion. The torsional rigidity T of a cylindrical shaft with cross section Ω containing a neutral multicoated inclusion Σ is simply expressed by two parts T_Ω and T_i, one from the host shaft $\Omega \setminus \Sigma$ and the other from the multicoated inclusion Σ, given by

$$T\left(\Omega, \Sigma; \mu_0, \mu_1, \cdots, \mu_n\right) = T_\Omega + \frac{\pi}{2} R_1^4 \left[\sum_{k=1}^{n} \mu_k \left(r_{k-1}^4 - r_k^4\right) - \mu_0\right]. \tag{2}$$

In summary, neutral cylinders can serve as useful building bricks of an exactly solvable microgeometry for a host shaft under torsion. Depending on the cross-sectional shape of the host shaft, a neutral cylinder needs to be suitably multicoated. We have exactly identified the algebraic compatibility conditions between the shear rigidity of the host shaft, shear rigidities of the multicoated neutral inclusion and the constituent area fractions. In particular, we showed the key criterion to the neutrality can be expressed in terms of a continued fraction via a correspondence with the effective conductivity of multicoated composites. The exact torsional rigidity depends on the constituent shear rigidities, area fractions as well as the size distributions of constituent cylinders, but is independent of the assembly microstructure. This solution will serve as a benchmark for an analytical solution for the torsional rigidity of a cylindrical composite shaft. To our knowledge, this is the first microgeometry valid for arbitrary cross section of a cylindrical shaft that permits an exact determination of torsional rigidity.

References

[1] Chen, T., Benveniste, Y. & Chuang, P. C.: Exact solutions in torsion of composite bars: thickly coated neutral inhomogeneities and composite cylinder assemblages. *Proc. R. Soc. Lond.* A **458**: 1719-1759, 2002..

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