TWO-FLUID JETS AND WAKES

Patrick Weidman*, Andrzej Herczynski**, Georgy Burde***
*University of Colorado, Boulder, Colorado, 80309, USA
**Boston College, Chestnut Hill, Massachusetts, 02467 USA
***Ben Gurion University, Sede-Boker Campus, 84990 Israel

Summary Analytical solutions for laminar, horizontal, two-fluid jets and wakes are found in the boundary-layer approximation, using a nonstandard similarity solution *ansatz* to account for interface deflection in the presence of gravity. Planar and axisymmetric fan jets as well as classical and momentumless planar wakes are considered. All interfaces deflect from horizontal except the fan jet, and velocity profiles can be strongly asymmetrical.

The streamfunction formulation of the boundary layer equations for planar ($j = 0$) and radial ($j = 1$) laminar motion of fluid ($i = 1$) residing above fluid ($i = 2$) given by

$$(\psi_i)_y(\psi_i)_{xy} - (\psi_i)_x(\psi_i)_{yy} - jx^{-1}(\psi_i)_y^2 = \nu_i\, x^j\, (\psi_i)_{yyy} \tag{1}$$

form the basis for analyzing two-fluid jets and wakes along with (i) far-field conditions of quiescent or uniform flow, (ii) interfacial kinematic, normal and tangential stress conditions, and (iii) an appropriate integral constraint. The fluid densities and viscosities are ρ_i and ν_i and $\mu_i = \rho_i\nu_i$. Example applications are the two-fluid planar and axisymmetric jets formed by spreading water over the free surface of a slightly stratified liquid reservoir exposed to quiescent air; see the experiments of Didden and Maxworthy [1]. In each problem, the position of the interface is determined from the normal stress condition at the interface by assuming a non-standard similarity solution of the form

$$\psi_i = Ax^\alpha f_i(\eta_i), \qquad \eta_i = \frac{(y-\phi(x))}{B}\frac{1}{x^\beta} \tag{2}$$

for appropriate constants A and B with exponents α and β to be determined for each problem considered below.

PLANAR AND RADIAL FAN JETS

Planar and radial two-fluid jets must satisfy the integral constraint

$$J = x^j\left(\int_{-\infty}^{\phi(x)} \rho_2 u_2^2 dy + \int_{\phi(x)}^{\infty} \rho_1 u_1^2 dy\right) \tag{3}$$

where J is the total momentum flux in the two-flud system, $y = \phi(x)$ is the position of the two-fluid interface, and $u_i = x^{-j}(\psi_i)_y$. Solution of the jet problems yield streamwise sech^2-velocity profiles, asymmetric about the interface, and governed by the parameter $\chi = \rho_1\mu_1/\rho_2\mu_2$, isthe same as that found by Lock [2] for spatially-driven free shear flow. The normal stress condition reveals that the interface is flat for the fan jet ($j = 1$), but for the planar jet it is deflected according to

$$\phi(x) \approx J^{3/2}\frac{(\mu-1)}{g(1-\rho)}\left(\frac{1}{\chi(1+\chi^{1/2})}\right)^{2/3}\frac{1}{x^{4/3}} \tag{4}$$

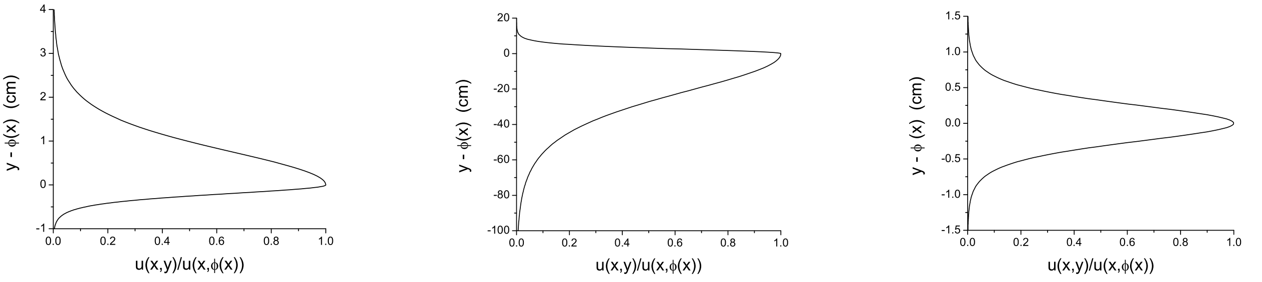
where $\rho = \rho_1/\rho_2$ and $\mu = \mu_1/\mu_2$. Thus the interface deflects either upward for $\mu > 1$ or downward for $\mu < 1$. The velocity profiles are symmetrically disposed about the interface only when $\chi = 1$. This shows that symmetric jets are possible for different fluids that possess a special relation among densities and viscosities. Sample velocity profiles for a planar jet are given in Figure 1.

The momentum flux in each fluid layer is partitioned according to $J_1 = \chi^{1/2}J/(1+\chi^{1/2})$ and $J_2 = J/(1+\chi^{1/2})$, yielding the momentum flux ratio $J_1/J_2 = \chi^{1/2}$. The jets penetrate differently into each fluid layer and their thickness grows according to $\delta_i(x) \sim x^{(2+j)/3}$.

CLASSICAL AND MOMENTUMLESS PLANAR WAKES

The same similarity *ansatz* (2) can be applied to the linearized formulation for the planar far-wake following Goldstein [3] for the classical wake formed by two-fluid flow over a stationary plate at the interface, and following Birkhoff and Zarantonello [4] for a momentumless wake over a self-propelled plate moving along the interface. The integral constraints for the classical and momentumless wakes are

$$D = U\left(\int_{-\infty}^{\phi(x)} \rho_2 w_2\, dy + \int_{\phi(x)}^{\infty} \rho_1 w_1\, dy\right); \qquad K = U\left(\int_{-\infty}^{\phi(x)} y^2\rho_2 w_2 dy + \int_{\phi(x)}^{\infty} y^2\rho_1 w_1 dy\right). \tag{5}$$



(a) (b) (c)

Figure 1: Normalized planar jet velocity profiles for (a) air-over-water, (b) air-over-oil and (c) water-over-water computed for jet momentum $J = 1000$ g/sec^2 at downstream position $x = 100$ cm.

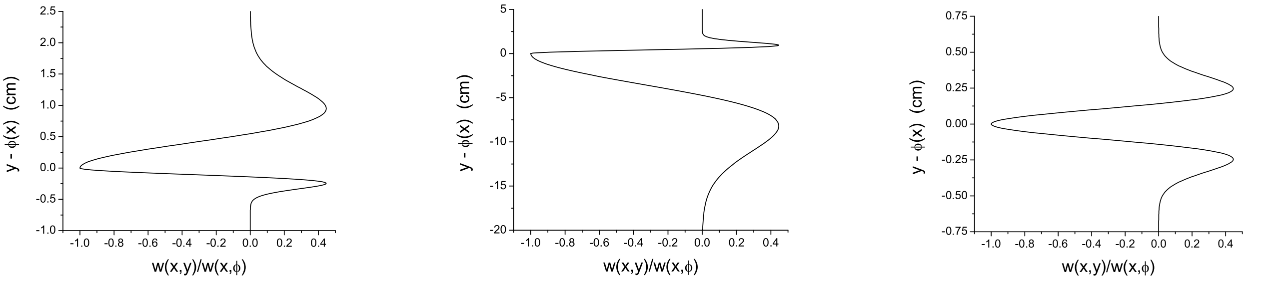
In both cases U is the uniform speed of the two-fluid flow and $w_i(x,y) = U - u_i(x,y)$ is the deficit wake velocity. For the classical wake D is the total plate friction drag, and for the momentumless wake K is the second moment of wake deficit momentum. The respective interface deflections are

$$\phi(x) \approx \frac{D}{gU^{1/2}} \left(\frac{\chi^{1/2}}{1+\chi^{1/2}} \right) \frac{1}{x^{3/2}}; \qquad \phi(x) \approx U^{1/2} K \, \frac{(\mu-1)}{g(1-\rho)} \left(\frac{1}{1+\Omega^{1/2}} \right) \frac{1}{x^{5/2}} \tag{6}$$

where χ is as defined for the two-fluid jets, and the new parameter governing the momentumless wake flow is $\Omega = \rho_1\mu_2^3/\rho_2\mu_1^3$. Note that the interface is always deflected upwards for the classical wake; for the momentumless wake the interface deflection is upward for $\mu > 1$ and downward for $\mu < 1$, just as for the two-fluid jet.

For the classical wake, the momentum deficit fluxes are partitioned according to $J_1 = \chi^{1/2}D/(1+\chi^{1/2})$ and $J_2 = D/(1+\chi^{1/2})$. Thus the deficit momentum flux ratio $J_1/J_2 = \chi^{1/2}$ is identical to that for two-fluid jets. The classical and momentumless wakes penetrate differently into their respective fluid layers, and in both cases their thicknesses grow according to $\delta(x) \sim x^{1/2}$.

Sample normalized momentumless wake velocity deficit profiles are given in Figure 2 below.



(a) (b) (c)

Figure 2: Normalized momentumless wake velocity deficit profiles for (a) air-over-water, (b) air-over-oil and (c) water-over-water computed for jet momentum $U = 100$ cm/sec^2 at downstream position $x = 100$ cm.

References

[1] Didden, N., Maxworthy, T.: The viscous spreading of plane and axisymmetric gravity currents. *J. Fluid Mech.* **121:**27-42, 1982.
[2] Lock, R. C.: The velocity distribution in the laminar boundary layer between parallel streams. *Quart. J. Mech. Appl. Math.* **4:**42-57, 1951.
[3] Goldstein, S.: On the two-dimensional steady flow of a viscous fluid behind a solid body, Part I. *Proc. Roy. Soc.* **A142:**545-562, 1933.
[4] Birkhoff, G., Zarantonello, E. H.: Jets, Wakes and Cavities. Academic Press, New York, 1957.